

A_∞ -ALGEBRAS ASSOCIATED WITH CURVES AND RATIONAL FUNCTIONS ON $\mathcal{M}_{g,g}$

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ABSTRACT. We consider the natural A_∞ -structure on the Ext-algebra $\text{Ext}^*(G, G)$ associated with the coherent sheaf $G = \mathcal{O}_C \oplus \mathcal{O}_{p_1} \oplus \dots \oplus \mathcal{O}_{p_n}$ on a smooth projective curve C , where $p_1, \dots, p_n \in C$ are distinct points. We study the homotopy class of the product m_3 . Assuming that $h^0(p_1 + \dots + p_n) = 1$ we prove that m_3 is homotopic to zero if and only if C is hyperelliptic and the points p_i are Weierstrass points. In the latter case we show that m_4 is not homotopic to zero, provided the genus of C is > 1 . In the case $n = g$ we prove that the A_∞ -structure is determined uniquely (up to homotopy) by the products m_i with $i \leq 6$. Also, in this case we study the rational map $\mathcal{M}_{g,g} \rightarrow \mathbb{A}^{g^2-2g}$ associated with the homotopy class of m_3 . We prove that for $g \geq 6$ it is birational onto its image, while for $g \leq 5$ it is dominant. We also give an interpretation of this map in terms of tangents to C in the canonical embedding and in the projective embedding given by the linear series $|2(p_1 + \dots + p_g)|$.

INTRODUCTION

Let C be a smooth projective curve of genus g over an algebraically closed field \mathbb{k} . With any generator G of the derived category $D^b(C)$ of coherent sheaves on C one can associate an A_∞ -algebra of endomorphisms of G , which is basically the Ext-algebra $\text{Ext}^*(G, G)$ equipped with higher operations defined uniquely up to homotopy. More precisely, this construction uses a dg-enhancement of $D^b(C)$ and applies to it the homological perturbation theory developed originally in [8], [9], [12], with explicit formulas given in [23], [17]. Furthermore, this A_∞ -algebra determines the derived category $D^b(C)$ (see [16, Thm. 3.1]), and hence the curve C (at least, if either $\text{char}(\mathbb{k}) = 0$ or $g \neq 1$, see [11]).

One of the possible choices of a generator of $D^b(C)$ is $G = \mathcal{O}_C \oplus L$, where L is a line bundle of degree 1. In the case of an elliptic curve the corresponding A_∞ -algebra was explicitly computed in [26] (assuming $\mathbb{k} = \mathbb{C}$). Note also that in this case there exists an autoequivalence of $D^b(C)$ sending G to $\mathcal{O}_C \oplus \mathcal{O}_p$. Also, in genus 1 case Lekili and Perutz studied in [19] homotopy classes of minimal A_∞ -structures on $\text{Ext}^*(G, G)$ extending the natural double product. Their results imply that all nontrivial homotopy classes of such A_∞ -structures arise either from elliptic curves or from the nodal plane cubic (see [19, Prop. 9]). Also, any such A_∞ -structure is finitely determined, i.e., determined up to homotopy by a finite number of the products m_i (actually by m_i with $i \leq 8$).

In this paper we consider a partial extension of this picture to higher genus curves and to the case of generators of $D^b(C)$ of the form

$$G = \mathcal{O}_C \oplus \mathcal{O}_{p_1} \oplus \dots \oplus \mathcal{O}_{p_n}, \tag{0.0.1}$$

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where p_1, \dots, p_n are n distinct points on C ($n \geq 1$), such that $h^0(p_1 + \dots + p_n) = 1$ (in particular, $n \leq g$). We would like to study the A_∞ -structure on the corresponding Ext-algebra

$$E = E_{g,n} = \text{Ext}^*(G, G),$$

which, depends only on n and g as an associative algebra (however, the higher products depend on (C, p_1, \dots, p_n)). In the case $n = g$ this Ext-algebra looks particularly nice: it is generated over the $(g+1)$ -dimensional subalgebra spanned by the natural idempotents in $\text{Hom}(G, G)$, by the one-dimensional spaces $\text{Hom}(\mathcal{O}_C, \mathcal{O}_{p_i})$ and $\text{Ext}^1(\mathcal{O}_{p_i}, \mathcal{O}_C)$. Furthermore, the defining relations between these generators are monomial (see (1.2.1)). It was proved in [5] that any minimal A_∞ -structure on $E_{g,g}$ is finitely determined, more precisely, determined up to homotopy by m_i with $i \leq 6$. This follows from the vanishing of certain graded components of the Hochschild cohomology of $E_{g,g}$ (see Theorem 1.3.1). We give a simpler proof of this vanishing using a minimal resolution of $E_{g,g}$ from [1]. On the other hand, we show that the same vanishing does not hold for the algebra $E_{g,n}$ if $n < g$, and in this case an A_∞ -structure on $E_{g,n}$ is not determined by any fixed finite number of m_i (with a possibly exception of the case $g = 2, n = 1$, see Remark 1.3.2.2).

We also consider the following basic question about the A_∞ -structure on $E_{g,n}$ coming from (C, p_1, \dots, p_n) : whether it is equivalent to the one with $m_3 = 0$. The answer is if and only if C is hyperelliptic and the points p_1, \dots, p_n are Weierstrass points (see Theorem 2.6.1). Furthermore, we also show that if $g > 1$ then either m_3 or m_4 is always nontrivial. The main point in the proof is that the Hochschild cohomology class given by the triple product m_3 can be recovered from the triple Massey products for the complexes

$$\mathcal{O} \longrightarrow \mathcal{O}_{p_i} \xrightarrow{[1]} \mathcal{O}_{p_i} \xrightarrow{[1]} \mathcal{O} \quad (0.0.2)$$

(see Proposition 1.3.3 and Section 2.4). In the hyperelliptic case we also study a certain quadruple Massey product and use [21, Thm. 3.1] to connect it with m_4 .

In the case $n = g$ we compute the triple Massey products (0.0.2) in terms of canonical rational sections of some natural line bundles on the moduli spaces $\mathcal{M}_{g,g}$ of curves with g marked points. Considering rational monomials of these sections we get $g^2 - 2g$ rational functions on $\mathcal{M}_{g,g}$, i.e., a rational map

$$\bar{\alpha} : \mathcal{M}_{g,g} \rightarrow \mathbb{A}^{g^2-2g} \quad (0.0.3)$$

(see Section 3.2). Assuming that the characteristic is zero, we prove that for $g \geq 6$ this map is birational onto its image (see Theorem 3.2.1), while for $g \leq 5$ it is dominant (see Theorem 5.2.2). The main idea in the proof of the former result is to reconstruct a curve C from the multiplication table between certain rational functions with prescribed polar parts at $p_1, \dots, p_g \in C$ (see Section 4). We also observe that the above rational map extends to stable curves and make explicit calculations for rational irreducible nodal curves (see Section 4.2) To prove dominance for $g \leq 5$ we first calculate the tangent map (see Section 5). Then we again use explicit calculations for rational nodal curves.

It is interesting to note that our triple Massey products (0.0.2) have a nice geometric interpretation: they record positions of the tangent lines to C at p_i in the canonical embedding, as well as, for $n = g$, of the tangent lines to C at p_i in the projective embedding given by the linear system $|2(p_1 + \dots + p_g)|$. Equivalently, they can be related to the Wahl

maps (defined in [30]) for the line bundles ω_C and $\mathcal{O}(2(p_1 + \dots + p_g))$ evaluated at the marked points (see Section 3.3).

The interest in characterizing A_∞ -algebras of the form $\text{Ext}^*(G, G)$ is motivated by the homological mirror symmetry conjecture, extended to non-Calabi-Yau manifolds (see [14]). Note that one knows the homological mirror correspondence involving a higher genus curve on the symplectic side and a Landau-Ginzburg model on the B-side due to the works [27], [4]. The other half of the correspondence for the same mirror pair should involve the derived category $D^b(C)$, governed by the A_∞ -algebra $\text{Ext}^*(G, G)$. Thus, finding a characterization of A_∞ -structures on $E_{g,g}$ arising from curves would be a step towards establishing such a correspondence.

The paper is organized as follows. In Section 1 we perform the calculation of the relevant Hochschild cohomology of the algebras $E_{g,n}$ (mostly for $n = g$). Section 2 is devoted to Massey products. Here we compute the triple Massey products governing the Hochschild cohomology class of m_3 on $E_{g,n}$, and a certain quadruple Massey product related to m_4 . This allows us to characterize geometrically vanishing of m_3 (see Theorem 2.6.1). In Section 3 we study the Massey products (0.0.2) globally over the moduli space of curves and show how they lead to the rational map (0.0.3). Also, in Section 3.3 we discuss the connection with the tangent lines to C in the canonical embedding and in the embedding given by $|2(p_1 + \dots + p_g)|$ and with the corresponding Wahl maps. In Section 4 we prove that (0.0.3) is birational onto its image for $g \geq 6$. Finally, in Section 5 we compute that tangent map to (0.0.3) and show that it is dominant for $g \leq 5$. The Appendix contains GAP codes that we used to make explicit calculations needed for some proofs.

Notation and conventions. We work over a fixed ground field \mathbb{k} , which is assumed to be algebraically closed whenever we discuss geometry. By a curve we mean a projective connected curve over \mathbb{k} . By a divisor on a (not necessarily smooth) curve C we always mean a divisor supported on the smooth part of C . For such a divisor D we use the notation $h^i(D) = \dim_{\mathbb{k}} H^i(C, \mathcal{O}(D))$ for $i = 0, 1$. We use the similar notation $h^i(L)$ for a line bundle L . In a triangulated category we denote $\text{Hom}^n(X, Y) := \text{Hom}(X, Y[n])$ for $n \in \mathbb{Z}$. We also depict elements of $\text{Hom}^n(X, Y)$ by arrows $X \xrightarrow{[n]} Y$. For a morphism $f : X \rightarrow Y$ we often denote the morphism $f[n] : X[n] \rightarrow Y[n]$ simply by f . For a dg-category \mathcal{C} we denote the differentials on the Hom-spaces by ∂ . We denote by $H^*(\mathcal{C})$ (resp., $H^0(\mathcal{C})$) the category obtained by passing to cohomology (resp., 0th cohomology) in Hom-spaces of \mathcal{C} . For dg \mathcal{C} -modules M, N we denote by $\text{Hom}_{\mathcal{C}}(M, N)$ the space of morphisms in the dg-category of dg \mathcal{C} -modules. All our A_∞ -structures are assumed to be strictly unital. For a vector space V with a basis B , an element $b \in B$, and an element $w \in W$ in another vector space, we denote by $[b]^*w$ the linear map $V \rightarrow W$ sending b to w and $B \setminus b$ to zero. For a line bundle or a 1-dimensional vector space L we often abbreviate $L^{\otimes n}$ as L^n .

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1. HOCHSCHILD COHOMOLOGY

We refer to [15] for an introduction to A_∞ -algebras. Recall that an A_∞ -structure (m_i) on a vector space A is called *minimal* if $m_1 = 0$. In this case m_2 equips A with a structure of a graded associative algebra. Thus, fixing m_2 we can talk about minimal A_∞ -structures on a graded algebra A . It is well known that equivalence classes of such A_∞ -structures on A are controlled by the Hochschild cohomology $HH^*(A) = H^*(A, A)$. In particular, if we have two such structures (m_i) and (m'_i) with $m_i = m'_i$ for $i < n$ then $m'_n - m_n$ is a Hochschild n -cocycle of internal degree $2 - n$, whose triviality means that the structure (m'_i) can be changed by a homotopy in such a way that $m_i = m'_i$ for $i \leq n$ (see [24, Lem. 2.2]). Let us denote by $HH^i(A)_j$ the component of the i th Hochschild cohomology group of internal degree j . We deduce that the vanishing of the Hochschild cohomology $HH^i(A)_{2-i}$ for all $i > N$ implies that any minimal A_∞ -structure on A is determined by (m_i) with $i \leq N$ up to homotopy.

We would like to apply these principles to the Ext-algebra $E = E_{g,n}$ (described explicitly below). In the case $n = g$ the relevant Hochschild cohomology were studied in [5]. Here we present two results of this study: an explicit description of $HH^3(E)_{-1}$ and the vanishing of $HH^i(E, E)_{2-i}$ for large i (see Proposition 1.3.3 and Theorem 1.3.1(i) below). In addition, we will show that the latter property does not hold if $n < g$.

1.1. Algebras $E_{g,n}$. Let C be a projective curve over \mathbb{k} of arithmetic genus g , and let p_1, \dots, p_n be distinct smooth points on C such that $h^0(p_1 + \dots + p_n) = 1$. Then from the short exact sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_C(p_1 + \dots + p_n) \rightarrow \mathcal{O}_C(p_1 + \dots + p_n)/\mathcal{O}_C \rightarrow 0$$

we get the boundary homomorphism

$$\bigoplus_{i=1}^n H^0(C, \mathcal{O}(p_i)/\mathcal{O}) \simeq H^0(C, \mathcal{O}(p_1 + \dots + p_n)/\mathcal{O}) \rightarrow H^1(C, \mathcal{O}),$$

which is an embedding, since the map $H^0(C, \mathcal{O}) \rightarrow H^0(C, \mathcal{O}(p_1 + \dots + p_n))$ is an isomorphism by our assumption.

Let $\theta_i \in \text{Hom}(\mathcal{O}_C, \mathcal{O}_{p_i})$ and $\eta_i \in \text{Ext}^1(\mathcal{O}_{p_i}, \mathcal{O}_C)$ be generators of these one-dimensional spaces. Then

$$\psi_i = \theta_i \circ \eta_i \in \text{Ext}^1(\mathcal{O}_{p_i}, \mathcal{O}_{p_i})$$

is a generator of $\text{Ext}^1(\mathcal{O}_{p_i}, \mathcal{O}_{p_i})$, and the elements

$$\xi_i = \eta_i \circ \theta_i \in \text{Ext}^1(\mathcal{O}_C, \mathcal{O}_C) = H^1(\mathcal{O}_C)$$

for $i = 1, \dots, n$ are linearly independent. In the case $n < g$ we extend these to a basis (ξ_1, \dots, ξ_g) of $H^1(\mathcal{O}_C)$.

Thus, the algebra $E_{g,n} = \text{Ext}^*(G, G)$, where G is given by (0.0.1), has the \mathbb{k} -basis

$$\begin{aligned} e_{\mathcal{O}} &:= \text{id}_{\mathcal{O}}, \quad e_{\mathcal{O}_{p_i}} := \text{id}_{\mathcal{O}_{p_i}}, \quad \theta_i, \quad \eta_i, \quad \psi_i, \quad i = 1, \dots, n; \\ \xi_j, \quad j &= 1, \dots, g. \end{aligned}$$

The only nontrivial products in $E_{g,n}$ are the obvious relations involving the idempotents $e_{\mathcal{O}}$ and $e_{\mathcal{O}_{p_i}}$, as well as the relations $\theta_i \eta_i = \psi_i$ and $\eta_i \theta_i = \xi_i$ for $i = 1, \dots, n$. In particular, this algebra does not depend on a specific curve and points on it.

Note that the algebra $E_{g,n}$ is the quotient-algebra of the path algebra of the quiver $\Gamma_{n,g}$ with $n+1$ vertices, marked with \mathcal{O} and \mathcal{O}_{p_i} , $i = 1 \dots, n$. The arrows in $\Gamma_{n,g}$ go in the direction opposite to the direction of morphisms in $D^b(C)$. Namely, for each p_i we have one arrow of degree 1 from \mathcal{O} to \mathcal{O}_{p_i} and one arrow of degree 0 in the opposite direction. In addition, we have $g-n$ loops of degree 1 at \mathcal{O} (that correspond to the generators ξ_{n+1}, \dots, ξ_g).

We denote by $E_{g,n}^+$ the ideal in $E_{g,n}$ obtained from paths of length ≥ 1 . In other words, this is the \mathbb{k} -subspace spanned by all θ_i , η_i , ψ_i and ξ_j .

1.2. Minimal resolution of $E_{g,g}$. Our method of calculating the Hochschild cohomology of $E = E_{g,g}$ is similar to that used in [19] for $g = 1$. Namely, we view E as the quotient of the path algebra $\mathbb{k}[\Gamma_{g,g}]$ by the monomial relations

$$\theta_i \eta_i \theta_i = \eta_i \theta_i \eta_i = \theta_i \eta_j = 0 \quad (1.2.1)$$

for $1 \leq i, j \leq g$, $i \neq j$. Hence, we can use a minimal projective resolution

$$\dots \rightarrow P_1 \rightarrow P_0 \rightarrow E$$

over the enveloping algebra $E^e = E \otimes E^{op}$ constructed in [1]. Let us recall this construction.

For every pair of vertices v, v' in the quiver we have the projective $E - E$ -bimodule

$$P_{v,v'} := E e_v \otimes e_{v'} E,$$

where e_v is the idempotent in E corresponding to v . For a path p in $\Gamma_{g,g}$ let v and v' be vertices such that $e_v p e_{v'} = p$ in $\mathbb{k}[\Gamma_{g,g}]$. Then we call $P_{v,v'}$ the *projective bimodule generated by $[p]$* , and denote its elements by $x[p]y$, where $x \in E e_v$, $y \in e_{v'} E$. We define the projective bimodule generated by a collection of paths as the direct sum of the projective bimodules generated by each path.

The $E - E$ -bimodules in our minimal resolution are defined as follows: $P_0 = E^e$, and for $j > 0$ we define P_j as the projective bimodule generated by the set $AP(j)$ of paths in $\Gamma_{g,g}$, defined by the following recursive procedure¹. By definition, $AP(1)$ consists of all paths of length 1, i.e., of θ_i and η_i ($i = 1, \dots, g$), while $AP(2)$ is exactly the set R of generating relations, namely of the paths in (1.2.1). Next, $AP(3)$ consists of paths “linking” pairs from R , namely,

$$AP(3) = \{(\theta_i \eta_i)^2, (\eta_i \theta_i)^2, \theta_i \eta_i \theta_i \eta_j, \theta_i \eta_j \theta_j \eta_j \mid 1 \leq i, j \leq g, i \neq j\}.$$

Let us denote by S the set of nonempty proper subwords in R . Thus,

$$S = \{\theta_i \eta_i, \eta_i \theta_i, \theta_i, \eta_i \mid 1 \leq i \leq g\}.$$

Note that each path in $AP(3)$ has form $p = sr$, where $r \in R$ and $s \in S$. Similarly, every path in $AP(j)$ will be of the form $p = sp'$, where $p' \in AP(j-1)$ and $s \in S$. By definition, for $j \geq 3$, $AP(j+1)$ is obtained by taking all paths $p = sp' \in AP(j)$ (where $p' \in AP(j-1)$, $s \in S$) and replacing s either by an element of R ending with s or, in the case $s = \eta_i \theta_i$, by $\theta_j \eta_i \theta_i$. For example,

$$AP(4) = \{(\theta_i \eta_i)^3, (\eta_i \theta_i)^3, (\eta_i \theta_i)^2 \eta_j, \theta_i \eta_i \theta_i \eta_j \theta_j \eta_j, \theta_i (\eta_j \theta_j)^2 \mid 1 \leq i, j \leq g, i \neq j\}. \quad (1.2.2)$$

¹We specialize a more general procedure from [1] to our situation

For a path $p = sp' \in AP(j)$ with $p' \in AP(j-1)$ and $s \in S$, we call s the *head* of p .

For an arrow a in the quiver $\Gamma_{g,g}$ we denote by $s(a)$ and $t(a)$ the source and the target of a (these are vertices of $\Gamma_{g,g}$). The first two of the differentials $d_j : P_j \rightarrow P_{j-1}$ are described as follows:

$$d_1 : [a] \mapsto e_{s(a)} \otimes a - a \otimes e_{t(a)},$$

$$d_2 : [a_1 a_2 \dots] \mapsto [a_1] a_2 \dots + a_1 [a_2] \dots + \dots,$$

where $a \in AP(1)$, $a_1 a_2 \dots \in AP(2)$. For odd $j > 2$ the differential is

$$d_j : [p] \mapsto s'[p'] - [p'']s'',$$

where we write $p \in AP(j)$ in the form $p = s'p' = p''s''$ with $p', p'' \in AP(j-1)$ and $s, s' \in S$. For even $j > 2$ the differential is

$$d_j : [p] \mapsto \sum s_1[p']s_2,$$

where $p \in AP(j)$ and the sum is over all decompositions $p = s_1 p' s_2$ with $p' \in AP(j-1)$. It is shown in [1, Thm. 4.1] that we get in this way a minimal projective resolution of E over E^e .

Lemma 1.2.1. (a) *The maximal internal degree of the generators of P_j is equal to*

$$h(j) := \begin{cases} j - [j/4] - 1, & j \equiv -1 \pmod{4}; \\ j - [j/4], & \text{otherwise.} \end{cases}$$

(b) *The maximal internal degree of the generators of P_{10} (resp., P_9) that end with θ_i is equal to 7 (resp., 6).*

Proof. (a) For each $s \in S$ let us denote by $a_j(s)$ the maximal degree of a word in $AP(j)$ with the head s (where $\deg \theta_i = 0$, $\deg \eta_i = 1$). Then from the definition of $AP(j)$ we get that $a_3(s) = 2$ for all s , as well as the recursive formulas

$$a_{j+1}(\theta\eta) = a_j(\theta) + 1,$$

$$a_{j+1}(\eta) = a_j(\theta\eta) + 1,$$

$$a_{j+1}(\eta\theta) = a_j(\eta) + 1,$$

$$a_{j+1}(\theta) = \max(a_j(\eta), a_j(\eta\theta))$$

for $j \geq 3$. Here we omit indices with θ and η since the value of $a_j(\cdot)$ does not depend on them. Now it is easy to check by induction that

$$a_j(\eta\theta) = h(j), \quad a_j(\eta) = h(j+1) - 1, \quad a_j(\theta\eta) = h(j+2) - 2, \quad a_j(\theta) = h(j+3) - 3,$$

which implies the assertion since $h(j+1) \leq h(j) + 1$.

(b) For $s \in S$ let us denote by $b_j(s)$ the maximal degree of a word in $AP(j)$ that has the head s and ends with θ_i (in the case when there are no such words we set $b_j(s) = -\infty$). These numbers satisfy the same recursive formulas as the numbers $a_j(s)$. From this we get

$$b_9(\theta) = b_9(\theta\eta) = b_9(\eta) = 6, \quad b_9(\eta\theta) = -\infty,$$

$$b_{10}(\theta) = 6, \quad b_{10}(\theta\eta) = b_{10}(\eta) = b_{10}(\eta\theta) = 7,$$

which implies our claim. \square

1.3. Calculations. Hochschild cohomology groups $HH^i(E_{g,g})_{2-i}$ were calculated in [5]. Here, using a minimal projective resolution of $E_{g,g}$ over its enveloping algebra, we give a different proof of the fact that these groups vanish for large i .

Theorem 1.3.1. (i) One has $HH^i(E_{g,g})_{2-i} = 0$ for $i > 8$. If $g > 1$ then $HH^i(E_{g,g})_{2-i} = 0$ for $i > 6$.

(ii) Assume $1 \leq n < g$. Then $HH^i(E_{g,n})_{2-i} \neq 0$ for all $i \geq 5$.

Proof. (i) We can compute the Hochschild cohomology of $E = E_{g,g}$ using the minimal resolution $P_\bullet \rightarrow E$ from Section 1.2. First, we claim that $\text{Hom}_{E^e}(P_i, E(2-i)) = 0$ for $i > 10$. Indeed, Lemma 1.2.1(a) implies that the internal degrees of generators of P_i are $< i - 2$. Next, for $i = 9$ or 10 we still claim that $\text{Hom}_{E^e}(P_i, E(2-i)) = 0$. Indeed, first we observe that in this case $a_i(s) \geq i - 2$ only when s begins with some η_k . Thus, the only possibly nontrivial morphism $P_i \rightarrow E(2-i)$ should send a generator $p \in AP(i)$ of degree $i - 2$, beginning with η_k , to E_0 . But any nonzero homogeneous element of degree 0 in E_0 that begins at the vertex \mathcal{O} , is proportional to $e_{\mathcal{O}}$, so p has to end with θ_j . By Lemma 1.2.1(b), this contradicts p being of degree $i - 2$.

Now assume that $g > 1$. We claim that the maps

$$d_9^* : \text{Hom}_{E^e}(P_8, E(-6)) \rightarrow \text{Hom}_{E^e}(P_9, E(-6)) \quad \text{and}$$

$$d_8^* : \text{Hom}_{E^e}(P_7, E(-5)) \rightarrow \text{Hom}_{E^e}(P_8, E(-5))$$

are injective and hence $HH^8(E)_{-6} = HH^7(E)_{-5} = 0$. First, let us analyze the spaces $\text{Hom}_{E^e}(P_8, E(-6))$ and $\text{Hom}_{E^e}(P_7, E(-5))$ using methods of Lemma 1.2.1. Since $(P_8)_{>6} = 0$, the only nonzero components of $\text{Hom}_{E^e}(P_8, E(-6))$ correspond to generators of degree 6 in P_8 mapping to E_0 . Among such generators $p \in AP(8)$ beginning with η_i we are only interested in those that end with some θ_j (otherwise, there is no element in E_0 to map p to). Note that $a_8(\theta) = b_8(\theta) = b_8(\theta\eta) = 5$ and $b_8(\eta) = -\infty$. In particular, we only need to consider p with the heads $\eta_i\theta_i$ or $\theta_i\eta_i$. In the former case p should end with some θ_j , so from the definition of $AP(\cdot)$ we see that $p = (\eta_i\theta_i)^3 p'$, where $p' \in AP(4)$ has the head $\eta_j\theta_j$ and ends with some θ_k . From the list (1.2.2) we conclude that $p' = (\eta_j\theta_j)^3$. On the other hand, in the case when p has the head $\theta_i\eta_i$ we have $p = \theta_i\eta_i\theta_i\eta_j\theta_j\eta_j p'$, where $p' \in AP(4)$ has the head $\theta_j\eta_j$ and ends with η_i (otherwise there is no element in E_0 to map p to). This gives either $p' = (\theta_i\eta_i)^3$ or $p' = \theta_j\eta_j\theta_j\eta_i\theta_i\eta_i$. Thus, $\text{Hom}_{E^e}(P_8, E(-6))$ has the following basis:

$$\begin{aligned} \alpha_1(i) &= [(\eta_i\theta_i)^6]^* e_{\mathcal{O}}, \\ \alpha_2(i, j) &= [(\eta_i\theta_i)^3(\eta_j\theta_j)^3]^* e_{\mathcal{O}}, \quad i \neq j, \\ \alpha_3(i) &= [(\theta_i\eta_i)^6]^* e_{\mathcal{O}_{p_i}}, \\ \alpha_4(i, j) &= [\theta_i\eta_i\theta_i(\eta_j\theta_j)^3\eta_i\theta_i\eta_i]^* e_{\mathcal{O}_{p_i}}, \quad i \neq j, \end{aligned} \tag{1.3.1}$$

Here we identify $\text{Hom}_{E^e}(P_8, E(-6))$ with the subspace of graded linear maps from the vector space with the basis $AP(8)$ to $E(-6)$ and denote by $[p]^* x$ the linear map that sends $p \in AP(8)$ to x and sends other basis elements to zero. To show that the images of the basis elements (1.3.1) under d_9^* stay linearly independent it is enough to give some basis elements $\beta_1(i), \dots, \beta_4(i, j)$ in $\text{Hom}_{E^e}(P_9, E(-6))$, such that $\beta_m(A)$ appears in $d_9^*(\alpha_m(A))$

but not in $d_9^*(\alpha_n(?))$ with $n > m$ and not in $d_9^*(\alpha_m(A'))$ with $A' \neq A$. For this purpose we take

$$\begin{aligned}\beta_1(i) &= [\theta_j(\eta_i\theta_i)^6]^*\theta_j, \\ \beta_2(i, j) &= [\theta_j(\eta_i\theta_i)^3(\eta_j\theta_j)^3]^*\theta_j, \quad i \neq j, \\ \beta_3(i) &= [\eta_i(\theta_i\eta_i)^6]^*\eta_i, \\ \beta_4(i, j) &= [(\eta_i\theta_i)^2(\eta_j\theta_j)^3\eta_i\theta_i\eta_i]^*\eta_i, \quad i \neq j,\end{aligned}$$

where in the first line we choose any j , different from i .

The proof of injectivity of d_8^* is very similar, where we use the basis of $\text{Hom}_{E^e}(P_7, E(-5))$ given by

$$\begin{aligned}\alpha_1(i) &= [(\eta_i\theta_i)^5]^*e_{\mathcal{O}}, \\ \alpha_2(i, j) &= [(\eta_j\theta_j)^3(\eta_i\theta_i)^2]^*e_{\mathcal{O}}, \quad i \neq j, \\ \alpha_3(i, j) &= [(\eta_j\theta_j)^2(\eta_i\theta_i)^3]^*e_{\mathcal{O}}, \quad i \neq j, \\ \alpha_4(i) &= [(\theta_i\eta_i)^5]^*e_{\mathcal{O}_{p_i}}, \\ \alpha_5(i, j) &= [\theta_i\eta_i\theta_i(\eta_j\theta_j)^2\eta_i\theta_i\eta_i]^*e_{\mathcal{O}_{p_i}}, \quad i \neq j, \\ \alpha_6(i, j) &= [\theta_i\eta_i\theta_i(\eta_j\theta_j)^3\eta_i]^*e_{\mathcal{O}_{p_i}}, \quad i \neq j, \\ \alpha_7(i, j) &= [\theta_i(\eta_j\theta_j)^3\eta_i\theta_i\eta_i]^*e_{\mathcal{O}_{p_i}}, \quad i \neq j\end{aligned}$$

and the following basis elements in $\text{Hom}_{E^e}(P_8, E(-5))$:

$$\begin{aligned}\beta_1(i) &= [\theta_j(\eta_i\theta_i)^5]^*\theta_j, \\ \beta_2(i, j) &= [\theta_j(\eta_j\theta_j)^3(\eta_i\theta_i)^2]^*\theta_j, \quad i \neq j, \\ \beta_3(i, j) &= [\theta_i(\eta_j\theta_j)^2(\eta_i\theta_i)^3]^*\theta_i, \quad i \neq j, \\ \beta_4(i) &= [(\theta_i\eta_i)^6]^*\psi_i, \\ \beta_5(i, j) &= [(\eta_i\theta_i)^2(\eta_j\theta_j)^2\eta_i\theta_i\eta_i]^*\eta_i, \quad i \neq j, \\ \beta_6(i, j) &= [(\eta_i\theta_i)^2(\eta_j\theta_j)^3\eta_i]^*\eta_i, \quad i \neq j, \\ \beta_7(i, j) &= [\theta_i\eta_i\theta_i(\eta_j\theta_j)^3\eta_i\theta_i\eta_i]^*\psi_i, \quad i \neq j,\end{aligned}$$

where in the first line we choose any j , different from i .

(ii) Let us consider the standard complex (C^\bullet, δ) computing the Hochschild cohomology of the algebra $E = E_{g,n}$. The basis of E as R -bimodule gives us a basis of each C^n of the form $[w]^*b$, where $b \in E$ is a basis element and w is a (composable) word of length n in basis elements in E_+ . Let us set $\xi = \xi_{n+1}$. We claim that for $i \geq 5$ the element

$$c = [\xi\eta_1\psi_1\theta_1\xi^{i-4}]^*\xi_1 + [\xi\xi_1\eta_1\theta_1\xi^{i-4}]^*\xi_1 \in C^i \quad (1.3.2)$$

is a cocycle giving a nontrivial cohomology class. Indeed,

$$\delta([\xi\eta_1\psi_1\theta_1\xi^{i-4}]^*\xi_1) = -\delta([\xi\xi_1\eta_1\theta_1\xi^{i-4}]^*\xi_1) = -[\xi\eta_1\theta_1\eta_1\theta_1\xi^{i-4}]^*\xi_1,$$

so $\delta(c) = 0$. On the other hand, note that the product of any two consecutive letters in the word $w = \xi\eta_1\psi_1\theta_1\xi^{i-4}$ is zero. Hence, for any basis element $[w']^*b \in C^{i-1}$, such that

$[w']^*\xi$ appears in $\delta([w]^*b)$, we have either $w = \xi w'$ and $\xi_1 = \xi b$ or $w = w'\xi$ and $\xi_1 = b\xi$. Since ξ_1 is not divisible by ξ either on left or right, this is impossible. \square

Remarks 1.3.2. 1. In the case $g = 1$ the spaces $HH^6(E_{1,1})_{-4}$ and $HH^8(E_{1,1})_{-6}$ are one-dimensional. Furthermore, (assuming $\text{char}(\mathbb{k}) \neq 2, 3$) any minimal A_∞ -structure on $E_{1,1}$ extending the natural m_2 is equivalent to the one for which $m_3 = m_4 = m_5 = 0$, and the Hochschild classes of m_6 and m_8 completely determine the A_∞ -structure up to an equivalence (see [19, Thm. 5]). In the case $g > 1$ any A_∞ -structure is determined by the products m_i with $i \leq 6$. However, the situation is more complicated since m_3 is usually nonzero (see Theorem 2.6.1 below). Furthermore, Theorem 3.2.1 below implies that an A_∞ -structure on $E_{g,g}$ arising from a generic curve of genus g with g points, is determined among such A_∞ -structures by m_3 alone. However, it is not clear whether every generic A_∞ -structure on $E_{g,g}$ arises geometrically for $g > 1$ (this is true for $g = 1$).

2. Assume that $1 \leq n < g$ and $g \geq 3$. Then for any $i \geq 5$ there exists a minimal A_∞ -structure on $E_{g,n}$ with standard m_2 , such that m_i gives a nontrivial Hochschild cohomology class, and $m_j = 0$ for $j \neq 2, i$. Indeed, we can define m_i by the slight modification of the formula (1.3.2):

$$m_i = [\xi\eta_1\psi_1\theta_1\xi^{i-4}]^*\xi_j + [\xi\xi_1\eta_1\theta_1\xi^{i-4}]^*\xi_j \in C^i$$

for any $j \neq 1, n+1$. Then the A_∞ -axiom is satisfied for (m_2, m_i) . Hence, in this case an A_∞ -structure on $E_{g,n}$ is not determined by any fixed finite number of (m_i) .

Proposition 1.3.3. Assume $\text{char}(\mathbb{k}) \neq 2$. Let us associate with a Hochschild 3-cocycle c on $E_{g,g}$ of internal degree -1 the constants $\alpha_{ij}(c)$ by

$$c(\eta_i, \psi_i, \theta_i) = \sum_j \alpha_{ij}(c)\xi_j.$$

Then the map

$$\alpha : c \mapsto (\alpha_{ij}(c))_{i \neq j} \tag{1.3.3}$$

induces an isomorphism of $HH^3(E_{g,g})_{-1}$ with the space of $g \times g$ -matrices with zeros on the diagonal.

Proof. Let us set $E = E_{g,g}$.

Step 1. First, we check that the map α is well-defined, i.e., that it vanishes on boundaries. Indeed, for a 2-cochain h of internal degree -1 we have $h(\eta_i, \psi_i) = \lambda \cdot \eta_i$ and $h(\psi_i, \theta_i) = \lambda' \cdot \theta_i$ for some constants λ, λ' . Hence,

$$(\delta h)(\eta_i, \psi_i, \theta_i) = -h(\eta_i, \psi_i) \cdot \theta_i - \eta_i \cdot h(\psi_i, \theta_i) = -\lambda \eta_i \theta_i - \lambda' \eta_i \theta_i = -(\lambda' + \lambda) \xi_i,$$

so $\alpha_{ij}(\delta h) = 0$ for $i \neq j$.

Step 2. Next, we claim that the map α is surjective. Indeed, for $i \neq j$ let us consider the Hochschild 3-cochain

$$f_{ij} = [\eta_i \psi_i \theta_i]^* \xi_j + [\eta_i \theta_i \xi_i]^* \xi_j.$$

It is easy to check that f_{ij} is a cocycle and that $\alpha(f_{ij})$ is the elementary matrix E_{ij} .

Step 3. $\dim HH^3(E)_{-1} = g(g-1)$. Using the minimal E^e -resolution $P_\bullet \rightarrow E$ from Section 1.2 we can identify the space $HH^3(E)_{-1}$ with the middle cohomology in

$$\text{Hom}_{E^e}(P_2, E(-1)) \xrightarrow{d_3^*} \text{Hom}_{E^e}(P_3, E(-1)) \xrightarrow{d_4^*} \text{Hom}_{E^e}(P_4, E(-1)).$$

First, note that $\text{Hom}_{E^e}(P_4, E(-1)) = 0$. Indeed, the only generators of degree ≤ 2 in P_4 correspond to paths $\theta_j \eta_i \theta_i \eta_i \theta_i \in AP(4)$, where $i \neq j$, and there are no elements of degree 1 in $e_{\mathcal{O}_{p_j}} E e_{\mathcal{O}}$. The space $\text{Hom}_{E^e}(P_3, E(-1))$ has the basis

$$[\eta_i \theta_i \eta_i \theta_i]^* \xi_j, [\theta_i \eta_i \theta_i \eta_i]^* \psi_i, \quad 1 \leq i, j \leq g.$$

On the other hand, the space $\text{Hom}_{E^e}(P_2, E(-1))$ has the basis

$$[\theta_i \eta_i \theta_i]^* \theta_i, [\eta_i \theta_i \eta_i]^* \eta_i, \quad i = 1, \dots, g.$$

Furthermore, the differential d_3^* is the direct sum of g copies of the same differential as for the $g = 1$ case, which is injective provided $\text{char}(\mathbb{k}) \neq 2$ (see the proof of [19, Thm. 4]). Hence, the dimension of the cohomology is $(g^2 + g) - 2g = g^2 - g$ as claimed. \square

2. MASSEY PRODUCTS

2.1. Massey products for dg categories. Let (A, ∂) be a dg-algebra over \mathbb{k} . For a ∂ -closed element $a \in A$ we denote by $[a]$ the corresponding cohomology class in $H^*(A) := H^*(A, \partial)$. Also for a homogeneous element $a \in A$ we set

$$\bar{a} = (-1)^{1+\deg(a)} a.$$

Suppose that we have a collection of homogeneous elements $a_\bullet = (a_{ij})$, where $0 \leq i < j \leq n$, $(i, j) \neq (0, n)$, satisfying the equations

$$\partial(a_{ij}) = \sum_{i < k < j} \bar{a}_{ik} a_{kj} \quad (2.1.1)$$

for all $0 \leq i < j \leq n$, $(i, j) \neq (0, n)$ (in particular, the elements $a_{i,i+1}$ are ∂ -closed). Then it is easy to check that

$$\mu(a_\bullet) := \sum_{0 < k < n} \bar{a}_{0k} a_{kn}$$

is also ∂ -closed. For given (homogeneous) cohomology classes $h_1, \dots, h_n \in H^*(A)$, one defines the n th Massey product

$$\langle h_1, \dots, h_n \rangle_{dg} \subset H^*(A)$$

as the subset formed by the classes $[\mu(a_\bullet)]$ as $a_\bullet = (a_{ij})$ runs through all collections as above with $[a_{i-1,i}] = h_i$, $i = 1, \dots, g$ (see [18], [22], [21]; we follow the sign convention of [21]). We call a collection a_\bullet as above a *defining system* for $\langle h_1, \dots, h_n \rangle_{dg}$. We say that the Massey product $\langle h_1, \dots, h_n \rangle_{dg}$ is defined if this subset is nonempty, i.e., there exists a defining system for $\langle h_1, \dots, h_n \rangle_{dg}$. For example, the double Massey product is always defined and is given by the usual product, up to a sign. The triple Massey product $\langle h_1, h_2, h_3 \rangle_{dg}$ is defined if and only if the double products $h_1 h_2$ and $h_2 h_3$ vanish in $H^*(A)$.

Now let \mathcal{C} be a dg-category over \mathbb{k} , and let $H^*(\mathcal{C})$ be the corresponding graded category obtained by passing to cohomology on morphisms. Let X_0, \dots, X_n be objects of \mathcal{C} . The equations (2.1.1) make sense for a collection of (homogeneous) morphisms $a_{ij} \in \text{Hom}_{\mathcal{C}}^*(X_j, X_i)$. Thus, similarly to the case of a dg-algebra one defines the Massey product of a collection (h_1, \dots, h_n) of homogeneous morphisms in $H^*(\mathcal{C})$, where $h_i \in H^* \text{Hom}_{\mathcal{C}}(X_i, X_{i-1})$.

Recall that the homological perturbation theory provides a minimal A_∞ -structure on $H^*(\mathcal{C})$ (see e.g., [23]). We will use the following important connection between the dg Massey products and this A_∞ -structure.

Theorem 2.1.1. ([21, Thm. 3.1]) *Consider a minimal A_∞ -structure on $H^*(\mathcal{C})$ obtained by homological perturbation theory. Let $(h_i \in H^* \text{Hom}_{\mathcal{C}}(X_i, X_{i-1}))$, $i = 1, \dots, n$ be homogeneous elements, such that the Massey product $\langle h_1, \dots, h_n \rangle_{dg}$ is defined. Then*

$$(-1)^b m_n(h_1, \dots, h_n) \in \langle h_1, \dots, h_n \rangle_{dg}, \quad \text{where} \\ b = 1 + \deg h_{n-1} + \deg h_{n-3} + \deg h_{n-5} + \dots$$

Recall that for a dg-category \mathcal{C} one also has the dg-category of (left) dg \mathcal{C} -modules $\mathcal{C} - \text{mod}$, i.e., of dg-functors from \mathcal{C} to the dg-category of \mathbb{k} -complexes. For a pair M_1, M_2 of dg \mathcal{C} -modules the obvious identification gives isomorphisms of complexes

$$\text{Hom}_{\mathcal{C}}^\bullet(M_1, M_2[1]) \simeq \text{Hom}_{\mathcal{C}}^\bullet(M_1, M_2)[1], \quad \text{Hom}_{\mathcal{C}}^\bullet(M_1[-1], M_2) \simeq \text{Hom}_{\mathcal{C}}^\bullet(M_1, M_2)[1]', \quad (2.1.2)$$

where for a complex (C, d) we denote by $C[1]'$ the complex $(C[1], d)$ (recall that the usual shifted complex $C[1]$ has the differential $-d$). We need to investigate the effect of shifts on the Massey products.

Lemma 2.1.2. *Let M_0, M_1, \dots, M_n be dg \mathcal{C} -modules, and let $\widetilde{M}_i = M_i[m_i]$ for some $m_i \in \mathbb{Z}$, $i = 0, \dots, n$. Let $h_i \in H^* \text{Hom}_{\mathcal{C}}(M_i, M_{i-1})$ be homogeneous elements, such that the Massey product $\langle h_1, \dots, h_n \rangle_{dg}$ is defined. Let $\tilde{h}_i \in H^* \text{Hom}_{\mathcal{C}}(\widetilde{M}_i, \widetilde{M}_{i-1})$ be an element corresponding to h_i under the obvious identification between the relevant spaces. Then one has*

$$\langle \tilde{h}_1, \dots, \tilde{h}_n \rangle_{dg} = (-1)^{m_1 + \dots + m_{n-1}} \langle h_1, \dots, h_n \rangle \subset H^* \text{Hom}_{\mathcal{C}}(\widetilde{M}_n, \widetilde{M}_0) \simeq H^* \text{Hom}_{\mathcal{C}}(M_n, M_0).$$

Proof. It is enough to consider the case when $m_i = 0$ for all $i \neq i_0$ and $m_{i_0} = 1$. Let $a_\bullet = (a_{ij})$ be a defining system for $\langle h_1, \dots, h_n \rangle_{dg}$, so that $[a_{i,i-1}] = h_i$. Set

$$\tilde{a}_{ij} = \begin{cases} -a_{ij}, & i < i_0 < j \\ a_{ij}, & \text{otherwise} \end{cases}.$$

Using isomorphisms (2.1.2), one can easily check that $\tilde{a}_\bullet = (\tilde{a}_{ij})$ is a defining system for $\langle \tilde{h}_1, \dots, \tilde{h}_n \rangle_{dg}$. Note that in the case $i_0 = 0$ or $i_0 = n$ we have $\tilde{a}_{ij} = a_{ij}$, so the two Massey products are the same. On the other hand, if $0 < i_0 < n$ then

$$\mu(\tilde{a}_\bullet) = -\mu(a_\bullet),$$

which implies the result. \square

Using Lemma 2.1.2 we can reduce the study of dg Massey products for the category of dg-modules to the case when all $a_{i,i-1}$ have degree 1 (and hence all a_{ij} in a defining system have degree 1). This will allow us to relate defining systems for Massey products to twisted complexes.

2.2. Convolutions in dg and triangulated categories. Let \mathcal{C} be a dg category. A *twisted complex*² over $\mathcal{C} - \text{mod}$ is a collection $\mathbf{M} = (M_i, a_{ij})$, where M_i , $i \in \mathbb{Z}$ are dg

²our convention on the numbering of M_i and degrees of a_{ij} differs from that of [2].

\mathcal{C} -modules with $M_i = 0$ for $i \gg 0$, and $a_{ij} : M_j \rightarrow M_i$, $i < j$, are morphisms of degree 1 satisfying

$$\partial(a_{ij}) + \sum_{i < k < j} a_{ij} a_{jk} = 0 \quad (2.2.1)$$

for all $i < j$. We define a convolution of $\mathbf{M} = (M_i, a_{ij})$ as the following \mathcal{C} -module:

$$\text{conv}(\mathbf{M}) := (\bigoplus_i M_i, \partial + A), \quad (2.2.2)$$

where ∂ is the usual differential on $\bigoplus_i M_i$ and $A = (a_{ij})$ is the upper-triangular endomorphism of $\bigoplus_i M_i$ with components a_{ij} . It is easy to check that $(\partial + A)^2 = 0$, so (2.2.2) is indeed a dg \mathcal{C} -module.

Let $f : M_1 \rightarrow M_0$ be a closed morphism of degree 1 in $\mathcal{C} - \text{mod}$. We can view this morphism as a twisted complex. Note that its convolution has form

$$\text{Cone}_{dg}(f) := \text{conv}(f) = (M_1 \oplus M_2, \partial_{M_1 \oplus M_2} + f).$$

We can also view f as a closed morphism $\tilde{f} : M_1[-1] \rightarrow M_0$ of degree 0, and $\text{Cone}_{dg}(f)$ can be identified with the standard cone of \tilde{f} .

The convolution of an arbitrary twisted complex can be obtained by iterating the cone operation.

Lemma 2.2.1. *Let $\mathbf{M} = (M_i, a_{ij})$ be a twisted complex over $\mathcal{C} - \text{mod}$, where $M_i \neq 0$ only for $i = 0, \dots, n$. Let us consider the truncated twisted complex $\tau_{[1,n]}\mathbf{M}$ obtained by considering only M_1, \dots, M_n and a_{ij} with $i \geq 1$. Then (a_{0i}) are components of a closed morphism of degree 1*

$$(a_{0\bullet}) : \text{conv}(\tau_{[1,n]}\mathbf{M}) \rightarrow M_0,$$

and there is a natural isomorphism of dg \mathcal{C} -modules

$$\text{conv}(\mathbf{M}) \simeq \text{Cone}_{dg}(a_{0\bullet}).$$

The proof is straightforward and is left to the reader.

Using the above lemma we can connect convolutions of twisted complexes over $\mathcal{C} - \text{mod}$ with convolutions in the triangulated category $H^0(\mathcal{C} - \text{mod})$. Recall (see [6, Exer. IV.2.1]) that a *convolution* of a complex

$$X_n \xrightarrow{d_n} X_{n-1} \rightarrow \dots \rightarrow X_1 \xrightarrow{d_1} X_0 \quad (2.2.3)$$

in a triangulated category \mathcal{T} (so that $d_i \circ d_{i+1} = 0$), is an object $T \in \mathcal{T}$ equipped with morphisms $\alpha : T \rightarrow X_n[n]$, $\beta : X_0 \rightarrow T$, such that there exists a diagram (called a *left Postnikov diagram*)

$$\begin{array}{ccccccc} & & X_{n-1} & \xrightarrow{d_{n-1}} & X_{n-2} & \xrightarrow{d_{n-2}} & \dots X_0 \\ & \nearrow d_n & & \searrow & \nearrow & & \searrow \beta \\ X_n = C_n & \xleftarrow{[1]} & C_{n-1} & \xleftarrow{[1]} & \dots C_1 & \xleftarrow{[1]} & C_0 = T \end{array}$$

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in which all the triangles (X_i, C_i, C_{i+1}) are distinguished, so that $\alpha : T \rightarrow X_n[n]$ is the composition of the arrows in the lower row.

Lemma 2.2.2. *Let (T, α, β) be a convolution of a complex (2.2.3). Then $(T[1], (-1)^n \alpha, \beta)$ is a convolution of the shifted complex*

$$X_n[1] \xrightarrow{d_n} X_{n-1}[1] \rightarrow \dots \rightarrow X_1[1] \xrightarrow{d_1} X_0[1].$$

Proof. This can be easily checked by induction in n using the fact that if

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is a distinguished triangle then the triangle

$$X[1] \xrightarrow{f} Y[1] \xrightarrow{g} Z[1] \xrightarrow{-h} X[2]$$

is also distinguished. \square

Lemma 2.2.3. *Let $\mathbf{M} = (M_i, a_{ij})$ be a twisted complex over $\mathcal{C} - \text{mod}$, where $M_i \neq 0$ only for $i = 0, \dots, n$. Consider the complex*

$$M_n[-n] \xrightarrow{d_n} M_{n-1}[-n+1] \rightarrow \dots \rightarrow M_1[-1] \xrightarrow{d_1} M_0 \quad (2.2.4)$$

in the triangulated category $H^0(\mathcal{C} - \text{mod})$, where $d_i = [a_{i-1,i}]$. Let $\pi : \text{conv}(\mathbf{M}) \rightarrow M_n = (M_n[-n])[n]$ and $\iota : M_0 \rightarrow \text{conv}(\mathbf{M})$ be the natural maps (given by the projection and the inclusion, respectively). Then the dg \mathcal{C} -module $\text{conv}(\mathbf{M})$, together with the maps $\alpha = (-1)^{\binom{n}{2}} \pi$ and $\beta = \iota$, is a convolution of the complex (2.2.4) in $H^0(\mathcal{C} - \text{mod})$.

Proof. We can proceed by induction in n (the case $n = 1$ was discussed before). Let $T = \text{conv}(\tau_{[1,n]}\mathbf{M})$. By induction assumption, $(T, (-1)^{\binom{n-1}{2}} \pi' : T \rightarrow M_n, \iota' : M_1 \rightarrow T)$ is a convolution of the complex

$$M_n[-n+1] \xrightarrow{d_n} M_{n-1}[-n+2] \rightarrow \dots \xrightarrow{d_2} M_1.$$

Hence, by Lemma 2.2.2, $(T[-1], (-1)^{\binom{n}{2}} \pi', \iota')$ is a convolution of

$$M_n[-n] \xrightarrow{d_n} M_{n-1}[-n+1] \rightarrow \dots \xrightarrow{d_2} M_1[-1].$$

On the other hand, by Lemma 2.2.1,

$$\text{conv}(\mathbf{M}) = \text{Cone}(\text{conv}(\tau_{[1,n]}\mathbf{M})[-1] \xrightarrow{a_0} M_0),$$

and the assertion follows. \square

2.3. Massey products for triangulated categories. Let us recall the definition of the Massey products for triangulated categories, sometimes called Toda brackets (see [3], [6, Exer. IV.2.3]). Let

$$X_n \xrightarrow{d_n} X_{n-1} \rightarrow \dots \rightarrow X_1 \xrightarrow{d_1} X_0 \quad (2.3.1)$$

be a complex in a triangulated category (so that $d_i \circ d_{i+1} = 0$). One defines $\langle d_1, \dots, d_n \rangle \subset \text{Hom}^{2-n}(X_n, X_0)$ as the set of all $p \circ q$, where we take a convolution $(T, \alpha : T \rightarrow X_{n-1}[n-2], \beta : X_1 \rightarrow T)$ of the complex $X_{n-1} \rightarrow \dots \rightarrow X_1$ (if it exists), and pick morphisms

$p : T \rightarrow X_0$ and $q : X_n \rightarrow T[2-n]$, such that d_n is equal to the composition $X_n \xrightarrow{q} T[2-n] \xrightarrow{\alpha} X_{n-1}$ and d_1 is equal to the composition $X_1 \xrightarrow{\beta} T \xrightarrow{p} X_0$.

It is well known that $0 \in \langle d_1, \dots, d_n \rangle$ is exactly the condition for the existence of a convolution of the complex (2.3.1). Also, for $\langle d_1, \dots, d_n \rangle$ to be non-empty it is necessary that $0 \in \langle d_1, \dots, d_{n-1} \rangle$ and $0 \in \langle d_2, \dots, d_n \rangle$ (and hence, the same is true for any proper substring in d_1, \dots, d_n).

It is easy to see that the triple product $\langle d_1, d_2, d_3 \rangle$ is non-empty provided $d_2 \circ d_3 = d_1 \circ d_2 = 0$, and is a coset for the subgroup

$$d_1 \circ \text{Hom}^{-1}(X_3, X_1) + \text{Hom}^{-1}(X_0, X_2) \circ d_3 \subset \text{Hom}^{-1}(X_3, X_0).$$

Note that a similar result holds for triple dg Massey products. The case of higher Massey products is more complicated. We will only consider a certain particular situation for the quadruple products in the triangulated and dg-settings (see Lemma 2.3.4 below).

The following relation between the Massey products in dg-categories and triangulated categories is well known to the experts and its various versions have appeared in the literature (see [28, Prop. 6.5] and [2, Sec. 5.A], which refers to the dissertation by Kapranov [13]).

Proposition 2.3.1. *Let \mathcal{C} be a dg-category, M_0, \dots, M_n a collection of dg \mathcal{C} -modules, and $d_i \in H^{k_i} \text{Hom}_{\mathcal{C}}(M_i, M_{i-1})$, $i = 1, \dots, n$, a collection of maps in $H^*(\mathcal{C} - \text{mod})$, such that the dg Massey product $\langle d_1, \dots, d_n \rangle_{dg}$ is defined. Then we have*

$$(-1)^{\sum_{i=1}^{n-1} (n-i)k_i} \langle d_1, \dots, d_n \rangle_{dg} \subset \langle d_1, \dots, d_n \rangle,$$

where on the right we consider the Massey product for the complex

$$M_n[-k_1 - \dots - k_n] \xrightarrow{d_n} M_{n-1}[-k_1 - \dots - k_{n-1}] \rightarrow \dots \rightarrow M_1[-k_1] \xrightarrow{d_1} M_0 \quad (2.3.2)$$

in the triangulated category $H^0(\mathcal{C} - \text{mod})$.

Proof. By Lemma 2.1.2, it is enough to consider the case when all $k_i = 1$. In this case we have to prove that

$$(-1)^{\binom{n-1}{2}} \langle d_1, \dots, d_n \rangle_{dg} \subset \langle d_1, \dots, d_n \rangle,$$

where on the right we consider the Massey product for the complex

$$M_n[-n] \xrightarrow{d_n} M_{n-1}[-n+1] \rightarrow \dots \xrightarrow{d_1} M_0.$$

Let $a_{\bullet} = (a_{ij})$, $a_{ij} \in \text{Hom}_{\mathcal{C}}^1(M_j, M_i)$ be a defining system for $\langle d_1, \dots, d_n \rangle_{dg}$, so that $[a_{i-1, i}] = d_i$ and (2.1.1) is satisfied with $\bar{a}_{ik} = a_{ik}$. Considering the restricted system $(a_{ij} \mid 1 \leq i, j \leq n-1)$ we obtain a twisted complex

$$\mathbf{M} = (M_{n-1}, \dots, M_1, (-a_{ij})_{1 \leq i < j \leq n-1}).$$

Let $T = \text{conv}(\mathbf{M})$ be the convolution of \mathbf{M} . By Lemma 2.2.3, $(T, (-1)^{\binom{n-2}{2}} \pi : T \rightarrow M_{n-1}, \iota : M_0 \rightarrow T)$ is a convolution of the complex

$$M_{n-1}[-n+2] \xrightarrow{d_{n-1}} \dots \xrightarrow{d_2} M_1$$

in the triangulated category $H^0(\mathcal{C} - \text{mod})$. Hence, by Lemma 2.2.2, $(T[-1], (-1)^{\binom{n-1}{2}}\pi, \iota)$ is a convolution of

$$M_{n-1}[-n+1] \xrightarrow{d_{n-1}} \dots \xrightarrow{d_2} M_1[-1].$$

Furthermore, we have closed morphisms of degree 1 of dg \mathcal{C} -modules

$$\tilde{q} = (a_{\bullet n}) : M_n \rightarrow T, \quad p = (a_{0\bullet}) : T \rightarrow M_0$$

(this follows from (2.1.1) for $j = n$ and $i = 0$, respectively), such that $\pi \circ \tilde{q} = a_{n-1,n}$ and $p \circ \iota = a_{01}$. Thus, the morphisms in $H^0(\mathcal{C} - \text{mod})$

$$q = (-1)^{\binom{n-1}{2}}\tilde{q} : M_n[-n] \rightarrow T[-n+1] = T[-1][2-n] \quad \text{and} \quad p : T[-1] \rightarrow M_0$$

satisfy the conditions in the definition of the Massey product $\langle d_1, \dots, d_n \rangle$. Since $p \circ q \in \langle d_1, \dots, d_n \rangle$ is represented by $(-1)^{\binom{n-1}{2}}\mu(a_{\bullet})$, the assertion follows. \square

On the other hand, by Theorem 2.1.1, the Massey product $\langle d_1, \dots, d_n \rangle_{dg}$ always contains $\pm m_n(d_1, \dots, d_n)$, where (m_{\bullet}) is a minimal A_{∞} -structure on $H^*(\mathcal{C})$ obtained by homological perturbation theory. This leads to the following result that will allow us to compute the Hochschild cohomology class of m_3 and, in a special situation, of m_4 , via the Massey products.

Corollary 2.3.2. *In the situation of Proposition 2.3.1 consider a minimal A_{∞} -structure on $H^*(\mathcal{C} - \text{mod})$ obtained by the homological perturbation theory. Assume that the Massey product $\langle d_1, \dots, d_n \rangle_{dg}$ is defined. Then*

$$(-1)^{b + \sum_{i=1}^{n-1} (n-i)k_i} m_n(d_1, \dots, d_n) \in \langle d_1, \dots, d_n \rangle$$

with $b = 1 + k_{n-1} + k_{n-3} + k_{n-5} + \dots$, where on the right we consider the Massey product for the complex (2.3.2) in $H^0(\mathcal{C} - \text{mod})$.

Remark 2.3.3. Any enhanced triangulated category in the sense of [2] can be realized as a full subcategory in $H^0(\mathcal{C} - \text{mod})$ for the corresponding dg-category \mathcal{C} (e.g., this follows from [2, Prop. 1.3, Prop. 3.2]). Therefore, we can apply Corollary 2.3.2 to compare the Massey products in an enhanced triangulated category with the higher products obtained by the homological perturbation theory.

Later we will need the following result.

Lemma 2.3.4. *(i) Suppose we have a complex (2.3.1) in a triangulated category, where $n = 4$. Assume that $0 \in \langle d_1, d_2, d_3 \rangle$, $0 \in \langle d_2, d_3, d_4 \rangle$, and a left Postnikov system for the complex*

$$X_3 \xrightarrow{d_3} X_2 \xrightarrow{d_2} X_1$$

is unique up to an isomorphism (identical on X_i). Then the Massey product $\langle d_1, d_2, d_3, d_4 \rangle$ is non-empty. Also, for any $\mu, \mu' \in \langle d_1, d_2, d_3, d_4 \rangle$ one has

$$\mu - \mu' \in \langle \text{Hom}^{-1}(X_2, X_0), d_3, d_4 \rangle + d_1 \circ \text{Hom}^{-2}(X_4, X_1). \quad (2.3.3)$$

(ii) Let \mathcal{C} be a dg-category, and let d_1, d_2, d_3, d_4 be a sequence of composable arrows in $H^0(\mathcal{C} - \text{mod})$ satisfying the assumptions of (i). Then the dg Massey product $\langle d_1, d_2, d_3, d_4 \rangle_{dg}$ is defined.

Proof. (i) By definition, $\langle d_1, d_2, d_3, d_4 \rangle$ consists of $p \circ q$, where p and q come from a diagram

$$\begin{array}{ccccccc}
 X_4 & \xrightarrow{d_4} & X_3 & \xrightarrow{d_3} & X_2 & \xrightarrow{d_2} & X_1 & \xrightarrow{d_1} & X_0 \\
 & & \searrow \begin{smallmatrix} [-1] \\ t \end{smallmatrix} & & \downarrow \iota & & \nearrow \begin{smallmatrix} [1] \\ \pi \end{smallmatrix} & & \\
 & & & & P & & & & \\
 & \searrow \begin{smallmatrix} [-2] \\ q \end{smallmatrix} & & & \uparrow \begin{smallmatrix} [1] \\ \epsilon \end{smallmatrix} & & \nearrow \begin{smallmatrix} \tilde{d}_2 \\ \delta \end{smallmatrix} & & \\
 & & & & T & & \nwarrow \begin{smallmatrix} p \end{smallmatrix} & &
 \end{array} \tag{2.3.4}$$

in which the triangles (X_3, X_2, P) and (P, X_1, T) are distinguished (so in the middle we have a left Postnikov system for $X_3 \rightarrow X_2 \rightarrow X_1$) and all other triangles are commutative. To show the existence of such a diagram we observe first that we can always construct a left Postnikov system in the middle and a morphism t such that $\pi \circ t = d_4$ (since $d_3 \circ d_4 = 0$). We have

$$\tilde{d}_2 \circ t \in \langle d_2, d_3, d_4 \rangle.$$

Hence, the assumption $0 \in \langle d_2, d_3, d_4 \rangle$ implies

$$\tilde{d}_2 \circ t = d_2 \circ f + g \circ d_4$$

for some $f \in \text{Hom}^{-1}(X_4, X_2)$ and $g \in \text{Hom}^{-1}(X_3, X_1)$. Thus, changing \tilde{d}_2 to $\tilde{d}_2 - g \circ \pi$ and t to $t - \iota \circ f$ we can achieve that

$$\pi \circ t = d_4, \quad \tilde{d}_2 \circ t = 0. \tag{2.3.5}$$

By the uniqueness of the left Postnikov diagram in the middle, in fact, the needed t exists for any choice of \tilde{d}_2 (since two such choices differ by an automorphism of P compatible with π and ι). Once we have t satisfying (2.3.5), we can find q such that $\epsilon \circ q = t$. On the other hand, the morphism p in the diagram exists provided $d_1 \circ \tilde{d}_2 = 0$. It is easy to see that

$$d_1 \circ \tilde{d}_2 = \mu \circ \pi$$

for some $\mu \in \langle d_1, d_2, d_3 \rangle$. Hence,

$$\mu = d_1 \circ g' + h \circ d_3$$

for some $g' \in \text{Hom}^{-1}(X_3, X_1)$ and $h \in \text{Hom}^{-1}(X_2, X_0)$. This implies that

$$d_1 \circ \tilde{d}_2 = d_1 \circ g' \circ \pi,$$

so changing \tilde{d}_2 by $\tilde{d}_2 - g' \circ \pi$ we will have $d_1 \circ \tilde{d}_2 = 0$, which will give the morphism p .

It remains to establish (2.3.3). By assumption, up to an isomorphism, two diagrams (2.3.4) differ only by a choice of the maps (t, p, q) . Given another diagram with maps (t', p', q') we can write

$$p' \circ q' - p \circ q = (p' - p) \circ q' + p \circ (q' - q).$$

Now we have $q' - q = \delta \circ x$ for some $x \in \text{Hom}^{-2}(X_4, X_1)$ and $p' - p = y \circ \epsilon$ for some $y \in \text{Hom}^{-1}(P, X_0)$. Therefore,

$$p' \circ q' - p \circ q = y \circ t + d_1 \circ x.$$

It remains to observe that $y \circ t \in \langle y \circ \iota, d_3, d_4 \rangle$.

(ii) Let $a_{i-1,i} \in \text{Hom}_{\mathcal{C}}^0(X_i, X_{i-1})$ be representatives of d_i for $i = 1, 2, 3, 4$. By assumption, there exist elements $a_{24} \in \text{Hom}_{\mathcal{C}}^{-1}(X_4, X_2)$ and $a_{13} \in \text{Hom}_{\mathcal{C}}^{-1}(X_3, X_1)$ such that

$$\partial(a_{24}) = -a_{23}a_{34}, \quad \partial(a_{13}) = -a_{12}a_{23}.$$

By Proposition 2.3.1, we have $0 \in \langle d_2, d_3, d_4 \rangle_{dg}$. Hence,

$$a_{13}a_{34} - a_{12}a_{24} = xa_{34} + a_{12}y + \partial(a_{14})$$

for some $a_{14} \in \text{Hom}_{\mathcal{C}}^{-2}(X_4, X_1)$, $x \in \text{Hom}_{\mathcal{C}}^{-1}(X_3, X_1)$ and $y \in \text{Hom}_{\mathcal{C}}^{-1}(X_4, X_2)$, such that $\partial(x) = 0$, $\partial(y) = 0$. Hence, changing a_{13} to $a_{13} - x$ and a_{24} to $a_{24} + y$ we can achieve that

$$a_{13}a_{34} - a_{12}a_{24} = \partial(a_{14}).$$

Similarly, from the condition $0 \in \langle d_1, d_2, d_3 \rangle$ we obtain that for some a'_{13} , a_{02} and a_{03} one has

$$\partial(a'_{13}) = -a_{12}a_{23}, \quad \partial(a_{02}) = -a_{01}a_{12},$$

$$a_{02}a_{23} - a_{01}a'_{13} = \partial(a_{03}).$$

Now we need to use our assumption on uniqueness of a Postnikov system, up to an isomorphism, to find a relation between a'_{13} and a_{13} . Let

$$P = \text{Cone}_{dg}(a_{23}) \in \mathcal{C} - \text{mod}$$

be the cone of a_{23} , viewed as a closed morphism of degree 1 from $X_3[1]$ to X_2 , so that we have a triangle of closed morphisms

$$X_3 \xrightarrow{a_{23}} X_2 \xrightarrow{\iota} P \xrightarrow{\pi} X_3[1]$$

that becomes distinguished in the triangulated category $H^0(\mathcal{C} - \text{mod})$. Our assumption on the uniqueness of a Postnikov system means that there exists a unique morphism $\tilde{d}_2 \in H^0 \text{Hom}_{\mathcal{C}}(P, X_1)$ such that $\iota \circ \tilde{d}_2 = d_2$ in $H^0 \text{Hom}_{\mathcal{C}}(X_2, X_1)$, up to an automorphism of P in $H^0(\mathcal{C} - \text{mod})$, compatible with the cone structure of P . We have two such morphisms \tilde{d}_2 , namely

$$\tilde{d}_2 = (-a_{12}, a_{13}) \text{ mod im}(\partial) \quad \text{and} \quad \tilde{d}'_2 = (-a_{12}, a'_{13}) \text{ mod im}(\partial).$$

Therefore, we have

$$\tilde{d}'_2 = \tilde{d}_2 \circ F \tag{2.3.6}$$

for some automorphism $F : P \rightarrow P$ in $H^0(\mathcal{C} - \text{mod})$, compatible with the cone structure of P . Any such automorphism has form

$$F = \text{id}_P - \iota f \pi \text{ mod im}(\partial)$$

for some closed element $f \in \text{Hom}_{\mathcal{C}}^{-1}(X_3, X_2)$. Hence, the condition (2.3.6) gives

$$a'_{13} = a_{13} + a_{12}f + ga_{23} + \partial(h),$$

where $g \in \text{Hom}_{\mathcal{C}}^{-1}(X_2, X_1)$, $\partial(g) = 0$, and $h \in \text{Hom}_{\mathcal{C}}^{-2}(X_3, X_1)$ (the term ga_{23} comes from the form of the differential on $\text{Hom}_{\mathcal{C}}(P, X_1)$). Now setting

$$a'_{02} = a_{02} - a_{01}g, \quad a'_{03} = a_{03} - a_{02}f + a_{01}h$$

we obtain that

$$(a_{01}, a_{12}, a_{23}, a_{34}, a'_{02}, a_{13}, a_{24}, a'_{03}, a_{14})$$

is a defining system for $\langle d_1, d_2, d_3, d_4 \rangle$. \square

2.4. Some triple Massey products on curves. Let C be a curve and $p \in C$ a smooth point. Let us denote by ξ_p the image of a generator of $H^0(\mathcal{O}(p)/\mathcal{O})$ under the connecting homomorphism $H^0(\mathcal{O}(p)/\mathcal{O}) \rightarrow H^1(\mathcal{O})$. We would like to study the map

$$\text{Ext}^1(\mathcal{O}_p, \mathcal{O}) \otimes \text{Ext}^1(\mathcal{O}_p, \mathcal{O}_p) \otimes \text{Hom}(\mathcal{O}, \mathcal{O}_p) \rightarrow H^1(C, \mathcal{O})/\langle \xi_p \rangle \quad (2.4.1)$$

given by the triple Massey product in $D^b(C)$ of the type

$$\mathcal{O}[-2] \rightarrow \mathcal{O}_p[-2] \rightarrow \mathcal{O}_p[-1] \rightarrow \mathcal{O}.$$

Note that such a Massey product is always non-empty since $\text{Ext}^1(\mathcal{O}, \mathcal{O}_p) = \text{Ext}^2(\mathcal{O}_p, \mathcal{O}) = 0$ and the ambiguity is exactly $\langle \xi_p \rangle \subset H^1(C, \mathcal{O})$, which is equal to the image of the composition map

$$\text{Ext}^1(\mathcal{O}_p, \mathcal{O}) \otimes \text{Hom}(\mathcal{O}, \mathcal{O}_p) \rightarrow H^1(C, \mathcal{O}).$$

It is also compatible with the map $-m_3$, obtained by the homological perturbation theory (see Corollary 2.3.2 and Remark 2.3.3).

Note that we have canonical bases in spaces $\text{Hom}(\mathcal{O}, \mathcal{O}_p)$, $\text{Ext}^1(\mathcal{O}_p, \mathcal{O}_p) \otimes T^*$ and $\text{Ext}^1(\mathcal{O}_p, \mathcal{O}) \otimes T^*$, where $T = T$ is the tangent line to C at p . By the definition of the Massey product, we have to consider the canonical extension

$$0 \rightarrow T^* \otimes_{\mathbb{k}} \mathcal{O}_p \xrightarrow{i} \mathcal{O}_{2p} \xrightarrow{\pi} \mathcal{O}_p \rightarrow 0$$

inducing a generator of $\text{Ext}^1(\mathcal{O}_p, \mathcal{O}_p)$. Then we should consider liftings $c : \mathcal{O} \rightarrow \mathcal{O}_{2p}$ and $d : \mathcal{O}_{2p} \rightarrow (T^*)^{\otimes 2} \otimes_{\mathbb{k}} \mathcal{O}[1]$ such that $\pi \circ c = 1 : \mathcal{O} \rightarrow \mathcal{O}_p$ and $i \circ d : T^* \otimes_{\mathbb{k}} \mathcal{O}_p \rightarrow (T^*)^{\otimes 2} \otimes_{\mathbb{k}} \mathcal{O}[1]$ is the canonical element represented by the extension

$$0 \rightarrow (T^*)^{\otimes 2} \otimes_{\mathbb{k}} \mathcal{O} \rightarrow (T^*)^{\otimes 2} \otimes_{\mathbb{k}} \mathcal{O}(p) \rightarrow T^* \otimes_{\mathbb{k}} \mathcal{O}_p \rightarrow 0.$$

Thus, we can take $c = 1 \in H^0(C, \mathcal{O}_{2p})$ and d to be the class of the extension

$$0 \rightarrow (T^*)^{\otimes 2} \otimes_{\mathbb{k}} \mathcal{O} \rightarrow (T^*)^{\otimes 2} \otimes_{\mathbb{k}} \mathcal{O}(2p) \rightarrow \mathcal{O}_{2p} \rightarrow 0. \quad (2.4.2)$$

Our Massey product is the coset of the composition $d \circ c : \mathcal{O} \rightarrow (T^*)^{\otimes 2} \otimes_{\mathbb{k}} \mathcal{O}[1]$ in $(T^*)^{\otimes 2} \otimes_{\mathbb{k}} H^1(\mathcal{O})/\langle \xi_p \rangle$. In other words, this is the image of $1 \in H^0(\mathcal{O}_{2p})$ under the boundary homomorphism

$$\delta_{2p} : H^0(\mathcal{O}_{2p}) \rightarrow (T^*)^{\otimes 2} \otimes_{\mathbb{k}} H^1(\mathcal{O}) \quad (2.4.3)$$

associated with the extension (2.4.2), viewed modulo $\langle \xi_p \rangle$. Since the latter subspace is the image under δ_{2p} of the subspace $H^0(T^* \otimes_{\mathbb{k}} \mathcal{O}_p) \subset H^0(\mathcal{O}_{2p})$, we obtain that our Massey product is zero if and only if

$$H^0(\mathcal{O}_{2p}) = \ker(\delta_{2p}) + H^0(T^* \otimes_{\mathbb{k}} \mathcal{O}_p).$$

Since $\ker(\delta_{2p})$ is the image of the homomorphism

$$(T^*)^{\otimes 2} \otimes_{\mathbb{k}} H^0(\mathcal{O}(2p)) \rightarrow H^0(\mathcal{O}_{2p}),$$

the Massey product vanishes if and only if the composed map

$$(T^*)^{\otimes 2} \otimes_{\mathbb{k}} H^0(\mathcal{O}(2p)) \rightarrow H^0(\mathcal{O}_{2p}) \rightarrow H^0(\mathcal{O}_{2p})/H^0(T^* \otimes_{\mathbb{k}} \mathcal{O}_p) \simeq H^0(\mathcal{O}_p)$$

is surjective. In other words, this is equivalent to surjectivity of the map

$$H^0(\mathcal{O}(2p)) \rightarrow H^0(\mathcal{O}(2p)/\mathcal{O}(p)),$$

or to the condition $H^0(C, \mathcal{O}(2p)) \not\subset H^0(C, \mathcal{O}(p))$. Thus, we obtain the following result.

Proposition 2.4.1. *Let C be a curve, $p \in C$ a smooth point. The Massey product (2.4.1) vanishes if and only if $H^0(C, \mathcal{O}(2p)) \not\subset H^0(C, \mathcal{O}(p))$. For example, if C is smooth and projective of genus $g \geq 1$ then this happens if and only if either $g = 1$ or C is hyperelliptic and p is a Weierstrass point of C .*

Next, we are going to compute the Massey product (2.4.1) in terms of an additional data allowing to represent classes in $H^1(\mathcal{O})$. Namely, let g be the arithmetic genus of C , and let us assume that D is an effective divisor of degree $g - 1$ (supported on the smooth part of C) such that $h^0(D + p) = 1$ and $p \notin \text{supp}(D)$. Then the boundary homomorphism

$$\delta_{D+p} : H^0(\mathcal{O}(D + p)/\mathcal{O}) \rightarrow H^1(\mathcal{O})$$

associated with an exact sequence $0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(D + p) \rightarrow \mathcal{O}(D + p)/\mathcal{O} \rightarrow 0$ is an isomorphism. Consider also the similar boundary homomorphism

$$\delta_{D+2p} : H^0(\mathcal{O}(D + 2p)/\mathcal{O}) \rightarrow H^1(\mathcal{O}),$$

so that δ_{D+p} is the restriction of δ_{D+2p} to the subspace $H^0(\mathcal{O}(D + p)/\mathcal{O}) \subset H^0(\mathcal{O}(D + 2p)/\mathcal{O})$. Note that the kernel of δ_{D+2p} is the image of the natural embedding

$$H^0(\mathcal{O}(D + 2p))/H^0(\mathcal{O}) \rightarrow H^0(\mathcal{O}(D + 2p)/\mathcal{O}).$$

Thus, for $x \in H^0(\mathcal{O}(D + 2p)/\mathcal{O})$ we can write

$$\delta_{D+2p}(x) = \delta_{D+p}(y)$$

where $y \in H^0(\mathcal{O}(D + p)/\mathcal{O})$ is such that $x \equiv y + s \pmod{\mathcal{O}}$ for some global section $s \in H^0(\mathcal{O}(D + 2p))$.

We want to compute the image of a generator of $H^0(\mathcal{O}(2p)/\mathcal{O})$ under the composition of the boundary homomorphism

$$\delta_{2p} : H^0(\mathcal{O}(2p)/\mathcal{O}) \rightarrow H^1(\mathcal{O}),$$

with the projection $H^1(\mathcal{O}) \rightarrow H^1(\mathcal{O})/\langle \xi_p \rangle$. Since δ_{2p} is just the restriction of δ_{D+2p} , we can apply the above recipe to $x \in H^0(\mathcal{O}(2p)/\mathcal{O}) \subset H^0(\mathcal{O}(D + 2p)/\mathcal{O})$. Note that $\langle \xi_p \rangle = \delta_{D+p}(H^0(\mathcal{O}(p)/\mathcal{O}))$, so we need to find $y \in H^0(\mathcal{O}(D + p)/\mathcal{O})$ and $s \in H^0(\mathcal{O}(D + 2p))$ such that

$$x \equiv y + s \pmod{\mathcal{O}}$$

and then view y modulo $H^0(\mathcal{O}(p)/\mathcal{O})$. In other words, we need to consider the projection of y to $H^0(\mathcal{O}(D + p)/\mathcal{O}(p)) \simeq H^0(\mathcal{O}(D)/\mathcal{O})$. Since the polar part of y near $\text{supp } D$ is opposite to that of s , we obtain the following formula for the Massey product (2.4.1).

Proposition 2.4.2. *With the above choice of divisor D let us consider the restriction maps*

$$r_{D+2p,p} : H^0(\mathcal{O}(D+2p))/H^0(\mathcal{O}) \xrightarrow{\sim} H^0(\mathcal{O}(D+2p)/\mathcal{O}(D+p)) \simeq T^{\otimes 2} \quad \text{and}$$

$$r_{D+2p,D} : H^0(\mathcal{O}(D+2p))/H^0(\mathcal{O}) \rightarrow H^0(\mathcal{O}(D)/\mathcal{O}).$$

Then the map (2.4.1) is equal to

$$-\overline{\delta_{D+p}} \circ r_{D+2p,D} \circ r_{D+2p,p}^{-1} : T^{\otimes 2} \rightarrow H^1(\mathcal{O})/\langle \xi_p \rangle,$$

where

$$\overline{\delta_{D+p}} : H^0(\mathcal{O}(D)/\mathcal{O}) \xrightarrow{\sim} H^0(\mathcal{O}(D+p)/\mathcal{O}(p)) \rightarrow H^1(\mathcal{O})/\langle \xi_p \rangle$$

is the isomorphism induced by δ_{D+p} .

Assume now that we are in the situation of Section 1.1 with $n = g$, so we have g distinct smooth points $p_1, \dots, p_g \in C$ such that $h^0(p_1 + \dots + p_g) = 1$, and the corresponding classes ξ_i , $i = 1, \dots, g$, form a basis in $H^1(C, \mathcal{O})$ (we use the notation from Section 1.1 for the basis elements in various Ext-spaces). Let T_{p_i} denote the tangent line to C at p_i . We have a natural isomorphism

$$T_{p_i} \simeq H^0(C, \mathcal{O}(p_i)/\mathcal{O}) \simeq \text{Ext}^1(\mathcal{O}_{p_i}, \mathcal{O}_{p_i}),$$

so we can think of ψ_i as a generator of T_{p_i} . Let us set $D_i = \sum_{j \neq i} p_j$.

Corollary 2.4.3. *The constants $\alpha_{ij}(m_3)$ associated with the natural A_∞ -structure on $E_{g,g}$ (see Proposition 1.3.3) can be computed as follows. Pick an element $\tilde{\psi}_i \in H^0(\mathcal{O}(2p_i + D_i))$ such that*

$$\tilde{\psi}_i \bmod \mathcal{O}(p_i + D_i) = (\psi_i)^{\otimes 2} \in H^0(\mathcal{O}(2p_i)/\mathcal{O}(p_i)) \simeq T_{p_i}^{\otimes 2}.$$

Then

$$\alpha_{ij}(m_3) \cdot \psi_j = \tilde{\psi}_i \bmod \mathcal{O}(2p_i + \sum_{k \neq i,j} p_k) \in H^0(\mathcal{O}(p_j)/\mathcal{O}) \simeq T_{p_j}.$$

Proof. This follows from the above computation of the Massey product $\langle \eta_i, \psi_i, \theta_i \rangle$ together with the compatibility

$$-m_3(\eta_i, \psi_i, \theta_i) \in \langle \eta_i, \psi_i, \theta_i \rangle$$

obtained from Corollary 2.3.2. □

Remark 2.4.4. The above Corollary shows that the constants $\alpha_{ij}(m_3)$ are related to a different kind of triple Massey product in $D^b(C)$ studied in [25]. Namely, setting $D = \sum_{k=1}^g p_k$ we have

$$\alpha_{ij}(m_3) \otimes \psi_j = \langle m_3(\mathcal{O}(D), p_i, p_j), \psi_i^{\otimes 2} \rangle, \quad (2.4.4)$$

where $m_3(L, x, y) \in (\omega_C \otimes L^{-1})|_x \otimes L|_y$ is the triple Massey product corresponding to the composable arrows

$$\mathcal{O}_C \rightarrow \mathcal{O}_x \xrightarrow{[1]} L \rightarrow \mathcal{O}_y$$

defined whenever $x \neq y$, x is a base point of $\omega_C \otimes L^{-1}$ and y is a base point of L (see [25, Sec. 1.1]). Note that in our case

$$m_3(\mathcal{O}(D), p_i, p_j) \in T_{p_i}^* \otimes \mathcal{O}(-D)|_{p_i} \otimes \mathcal{O}(D)|_{p_j} \simeq (T_{p_i}^*)^{\otimes 2} \otimes T_{p_j}.$$

The identity (2.4.4) also follows from the A_∞ -axioms associated with composable arrows

$$\mathcal{O} \rightarrow \mathcal{O}_{p_l} \xrightarrow{[1]} \mathcal{O}(D) \rightarrow \mathcal{O}_{p_k} \xrightarrow{[1]} \mathcal{O}$$

for $l = k$ and $l = i$. Picking one more point $q \in C$ generically we can write a formula for $m_3(\mathcal{O}(D), p_i, p_j)$ in terms of theta-functions. Namely, first one easily checks that $m_3(\mathcal{O}(D), p_i, p_j) = m_3(\mathcal{O}(D - q), p_i, p_j)$. Next, we represent $D - q$ as the sum of two divisors:

$$D - q = \xi + (D_{ij} + q),$$

where $\xi = p_i + p_j - 2q$ and $D_{ij} = \sum_{k \neq i, j} p_k$. We have a theta-function $\theta_{D_{ij}+q}$ on the Jacobian of C associated with the degree $g - 1$ divisor $D_{ij} + q$. Now by [25, Lem. 2.2] we obtain

$$m_3(\mathcal{O}(D), p_i, p_j) = m_3(\mathcal{O}(D - q), p_i, p_j) = -\frac{\theta_{D_{ij}+q}(2p_i - 2q)\theta'_{D_{ij}+q}(0)(p_i)}{\theta_{D_{ij}+q}(p_i - p_j)\theta_{D_{ij}+q}(p_i + p_j - 2q)},$$

where we view $\theta'_{D_{ij}+q}(0)$ as a global 1-form on C . Note that the formula of [25, Lem. 2.2] is applicable since p_i and p_j are not in the support of $D_{ij} + q$.

2.5. A quadruple Massey product. Next, let us assume that (C, p) is such that $H^0(C, \mathcal{O}(2p)) \not\subset H^0(C, \mathcal{O}(p))$ (e.g., C is a hyperelliptic smooth projective curve and p is a Weierstrass point).

Lemma 2.5.1. *Under the above assumption the quadruple Massey product in $D^b(C)$ of the type*

$$\mathcal{O}[-3] \rightarrow \mathcal{O}_p[-3] \rightarrow \mathcal{O}_p[-2] \rightarrow \mathcal{O}[-1] \xrightarrow{\xi_p} \mathcal{O}$$

gives rise to a well defined map

$$\langle \xi_p \rangle \otimes \text{Ext}^1(\mathcal{O}_p, \mathcal{O}) \otimes \text{Ext}^1(\mathcal{O}_p, \mathcal{O}_p) \otimes \text{Hom}(\mathcal{O}, \mathcal{O}_p) \rightarrow H^1(C, \mathcal{O}) / \langle \xi_p \rangle. \quad (2.5.1)$$

The corresponding dg Massey product (coming from some dg-enhancement of $D^b(C)$) is also defined.

Proof. We would like to apply Lemma 2.3.4 in our situation. Note that the relevant triple Massey products contain zero, since $\text{Hom}^2(\mathcal{O}_p, \mathcal{O}) = 0$ and the triple Massey product (2.4.1) vanishes by Proposition 2.4.1 (here we use our assumption on (C, p)). Next, we need to check the uniqueness of a left Postnikov diagram (up to an isomorphism) for the complex

$$\mathcal{O}_p \xrightarrow[\psi]{[1]} \mathcal{O}_p \xrightarrow[\eta]{[1]} \mathcal{O}.$$

Since the distinguished triangle containing ψ corresponds to a nontrivial extension

$$0 \rightarrow \mathcal{O}_p \xrightarrow{\iota} \mathcal{O}_{2p} \xrightarrow{\pi} \mathcal{O}_p \rightarrow 0,$$

it is enough to check that any diagram

$$\begin{array}{ccccc}
 \mathcal{O}_p & \xrightarrow[\psi]{[1]} & \mathcal{O}_p & \xrightarrow[\eta]{[1]} & \mathcal{O} \\
 & \nearrow \pi & \downarrow \iota & \nwarrow \tilde{\eta} & \\
 & & \mathcal{O}_{2p} & &
 \end{array}$$

in which the left triangle is distinguished, is obtained from any other such diagram by an automorphism of \mathcal{O}_{2p} . Indeed, two choices of $\tilde{\eta}$ differ by a morphism of the form $f \circ \pi$, where $f \in \text{Hom}^1(\mathcal{O}_p, \mathcal{O})$. Thus, f is a multiple of η : $f = c \cdot \eta$, and so

$$f \circ \pi = c(\eta \circ \pi) = \tilde{\eta} \circ (c \cdot \iota \circ \pi).$$

Now consider the automorphism $\text{id} + c(\iota \circ \pi)$ of \mathcal{O}_{2p} . This automorphism is compatible with the extension structure and sends $\tilde{\eta}$ to $\tilde{\eta} + f \circ \pi$, as required.

It remains to check that the ambiguity for our Massey product is exactly

$$\langle \xi_p \rangle \subset H^1(C, \mathcal{O}) = \text{Ext}^1(\mathcal{O}, \mathcal{O}).$$

By Lemma 2.3.4, we have to look at the composition $\xi_p \circ \text{Hom}(\mathcal{O}, \mathcal{O}) \subset \text{Ext}^1(\mathcal{O}, \mathcal{O})$ and at the triple Massey product

$$\langle \text{Ext}^1(\mathcal{O}_p, \mathcal{O}), \text{Ext}^1(\mathcal{O}_p, \mathcal{O}_p), \text{Hom}(\mathcal{O}, \mathcal{O}_p) \rangle \subset \text{Ext}^1(\mathcal{O}, \mathcal{O}).$$

Applying Proposition 2.4.1 one more time, we see that the latter product is $\langle \xi_p \rangle$.

The last assertion follows from Lemma 2.3.4(b). \square

By definition, the Massey product (2.5.1) is calculated as the composition $\tilde{\xi} \circ s$ in the following diagram

$$\begin{array}{ccccccc}
 \mathcal{O} & \longrightarrow & \mathcal{O}_p & \xrightarrow[\psi]{[1]} & \mathcal{O}_p & \xrightarrow[\eta]{[1]} & \mathcal{O} \xrightarrow[\xi_p]{[1]} \mathcal{O} \\
 & \searrow & \nearrow & \downarrow & \nwarrow & \nearrow & \\
 & & & \mathcal{O}_{2p} & & & \\
 & \searrow s & & \uparrow r & & \nearrow \tilde{\eta} & \\
 & & & \mathcal{O}(2p) & & \nwarrow \tilde{\eta} & \\
 & & & & & \nearrow \tilde{\eta} & \\
 & & & & & \nwarrow \tilde{\eta} &
 \end{array} \tag{2.5.2}$$

in which the triangle containing ψ and the triangle containing $\tilde{\eta}$ and r are distinguished and all the other triangles are commutative. Here r is the composition of the natural projection $\mathcal{O}(2p) \rightarrow \mathcal{O}(2p)/\mathcal{O}$ and an isomorphism $\mathcal{O}(2p)/\mathcal{O} \simeq \mathcal{O}_{2p}$. In other words, we pick an element $\tilde{\xi} \in H^1(\mathcal{O}(-2p))$ such that $t_*(\tilde{\xi}) = \xi_p$, where

$$t_* : H^1(\mathcal{O}(-2p)) \rightarrow H^1(\mathcal{O})$$

is the map induced by the canonical embedding $t : \mathcal{O}(-2p) \rightarrow \mathcal{O}$. On the other hand, we choose a section $s : \mathcal{O} \rightarrow \mathcal{O}(2p)$ such that $r(s) = 1 \in H^0(\mathcal{O}_{2p})$, and apply the induced map $s_* : H^1(\mathcal{O}(-2p)) \rightarrow H^1(\mathcal{O})$ to $\tilde{\xi}$. One can check directly that the ambiguities in the choices of $\tilde{\xi}$ and s do not change the coset of $s_*(\tilde{\xi})$ in $H^1(\mathcal{O})/\langle \xi_p \rangle$ (we also know this by Lemma 2.5.1). From this we obtain the following descriptions of the quadruple Massey product (2.5.1) similar to those for the triple Massey product (2.4.1).

Proposition 2.5.2. *Let C be a curve, $p \in C$ a smooth point, such that $H^0(C, \mathcal{O}(2p)) \not\subset H^0(C, \mathcal{O}(p))$. Let $T = T_p$ denote the tangent line to C at p .*

(a) *The Massey product (2.5.1) is given by the map*

$$\phi : T^{\otimes 2} \otimes \langle \xi_p \rangle \simeq T^{\otimes 3} \simeq H^0(\mathcal{O}(3p)/\mathcal{O}(2p)) \rightarrow H^1(\mathcal{O})/\langle \xi_p \rangle, \quad (2.5.3)$$

where the last arrow is induced by the boundary homomorphism $\delta_{3p} : H^0(\mathcal{O}(3p)/\mathcal{O}) \rightarrow H^1(\mathcal{O})$. The map ϕ vanishes if and only if $H^0(C, \mathcal{O}(3p)) \not\subset H^0(C, \mathcal{O}(2p))$.

(b) *Let g be the arithmetic genus of C , and let D be an effective divisor of degree $g - 1$ (supported on the smooth part of C) such that $h^0(D + p) = 1$ and $p \notin \text{supp}(D)$. Then we have*

$$\phi = -\overline{\delta_{D+p}} \circ r_{D+3p,D} \circ r_{D+3p,p}^{-1},$$

where

$$r_{D+3p,p} : H^0(\mathcal{O}(D + 3p))/H^0(\mathcal{O}(2p)) \xrightarrow{\sim} H^0(\mathcal{O}(3p)/\mathcal{O}(2p)) \simeq T^{\otimes 3},$$

$$r_{D+3p,D} : H^0(\mathcal{O}(D + 3p))/H^0(\mathcal{O}(2p)) \rightarrow H^0(\mathcal{O}(D)/\mathcal{O})$$

are natural restriction maps and $\overline{\delta_{D+p}}$ is the isomorphism from Proposition 2.4.2.

Proof. (a) Pick $s \in H^0(\mathcal{O}(2p))$ such that $\bar{s} = s \bmod \mathcal{O}(p) \neq 0$. Then, as we have seen above,

$$\phi(\bar{s} \otimes \xi_p) = s_*(t_*)^{-1}(\xi_p)$$

(the right-hand side is well defined in $H^1(\mathcal{O})/\langle \xi_p \rangle$). Recall that $\xi_p \in H^1(\mathcal{O})$ is the image of a generator ψ_p of $T \simeq H^0(\mathcal{O}(p)/\mathcal{O})$ under the boundary map $H^0(\mathcal{O}(p)/\mathcal{O}) \rightarrow H^1(\mathcal{O})$. Hence, if we pick a generator $\tilde{\psi}_p \in H^0(\mathcal{O}(p)/\mathcal{O}(-2p))$, such that $\tilde{\psi}_p \equiv \psi_p \bmod \mathcal{O}$, then $(t_*)^{-1}(\xi_p) \in H^1(\mathcal{O}(-2p))$ is represented modulo $\ker(t_*)$ by the image of $\tilde{\psi}_p$ under the boundary homomorphism

$$H^0(\mathcal{O}(p)/\mathcal{O}(-2p)) \rightarrow H^1(\mathcal{O}(-2p)).$$

Now the morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(-2p) & \longrightarrow & \mathcal{O}(p) & \longrightarrow & \mathcal{O}(p)/\mathcal{O}(-2p) & \longrightarrow & 0 \\ & & \downarrow s & & \downarrow s & & \downarrow s' & & \\ 0 & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{O}(3p) & \longrightarrow & \mathcal{O}(3p)/\mathcal{O} & \longrightarrow & 0 \end{array}$$

shows that $\phi(\bar{s} \otimes \xi_p)$ is represented by the image of $s'(\tilde{\psi}_p)$ under the boundary map $H^0(\mathcal{O}(3p)/\mathcal{O}) \rightarrow H^1(\mathcal{O})$, which implies our first assertion.

The map ϕ vanishes if and only if the image of the boundary homomorphism $\delta_{3p} : H^0(\mathcal{O}(3p)/\mathcal{O}) \rightarrow H^1(\mathcal{O})$ is equal to $\langle \xi_p \rangle$, which is the image of $\delta_{2p} : H^0(\mathcal{O}(2p)/\mathcal{O}) \rightarrow H^1(\mathcal{O})$. Since $H^0(\mathcal{O}(2p)/\mathcal{O})$ has codimension 1 in $H^0(\mathcal{O}(3p)/\mathcal{O})$, this happens exactly when $\ker(\delta_{2p})$ has codimension 1 in $\ker(\delta_{3p})$. But these kernels are $H^0(\mathcal{O}(2p))/H^0(\mathcal{O})$ and $H^0(\mathcal{O}(3p))/H^0(\mathcal{O})$, respectively, hence the assertion.

(b) Note that $\chi(D+p) = 1$, so $H^1(\mathcal{O}(D+p)) = 0$. It follows that $H^1(\mathcal{O}(D+2p)) = 0$, and hence, $h^0(D+2p) = \chi(D+2p) = 2$. Also, $h^0(p) \leq h^0(D+p) = 1$, so $h^0(p) = 1$ and $h^0(2p) = 2$. Therefore, the natural map $H^0(\mathcal{O}(2p)) \rightarrow H^0(\mathcal{O}(D+2p))$ is an isomorphism. Now the fact that $r_{D+3p,p}$ is an isomorphism follows from the long exact sequence of cohomology associated with the exact sequence

$$0 \rightarrow \mathcal{O}(D+2p) \rightarrow \mathcal{O}(D+3p) \rightarrow \mathcal{O}(3p)/\mathcal{O}(2p) \rightarrow 0.$$

We have a natural direct sum decomposition

$$H^0(\mathcal{O}(D+3p)/\mathcal{O}(p)) \simeq H^0(\mathcal{O}(D)/\mathcal{O}) \oplus H^0(\mathcal{O}(3p)/\mathcal{O}(p))$$

and a boundary map

$$\delta : H^0(\mathcal{O}(D+3p)/\mathcal{O}(p)) \rightarrow H^1(\mathcal{O}(p)) \simeq H^1(\mathcal{O})/\langle \xi_p \rangle.$$

We observe that the restriction of δ to the summand $H^0(\mathcal{O}(D)/\mathcal{O})$ is exactly the isomorphism $\overline{\delta_{D+p}}$, and the restriction of δ to $H^0(\mathcal{O}(3p)/\mathcal{O}(p))$ is compatible with ϕ . Now start with a section $s \in H^0(\mathcal{O}(D+3p))$ and write

$$s \bmod \mathcal{O}(p) = x + y$$

with $x \in H^0(\mathcal{O}(D)/\mathcal{O})$ and $y \in H^0(\mathcal{O}(3p)/\mathcal{O}(p))$. Then we have

$$\phi(r_{D+3p,p}(s)) = \delta(y),$$

$$\overline{\delta_{D+p}}(r_{D+3p,D}(s)) = \delta(x).$$

Since $\delta(x) + \delta(y) = \delta(s) = 0$, our assertion follows. \square

Corollary 2.5.3. *Let C be an irreducible projective curve with at most nodal singularities of arithmetic genus $g \geq 2$, and let $p \in C$ be a smooth point such that $H^0(C, \mathcal{O}(2p)) \not\subset H^0(C, \mathcal{O}(p))$. Then the Massey product (2.5.1) does not vanish.*

Proof. By Proposition 2.5.2, we have to check that $H^0(C, \mathcal{O}(3p)) = H^0(C, \mathcal{O}(2p))$. If C is smooth then the divisor $2p$ is in the hyperelliptic system, and the assertion follows easily. Thus, we can assume that C is singular. Let $\tilde{C} \rightarrow C$ be the normalization of C , so that C is obtained by gluing pairs of distinct points (a_i, b_i) , $i = 1, \dots, s$, on \tilde{C} . We denote by $p \in \tilde{C}$ the point corresponding to $p \in C$. If the genus of \tilde{C} is ≥ 2 then it is hyperelliptic and the assertion follows as in the smooth case.

Now assume that \tilde{C} has genus 1. The condition $h^0(C, \mathcal{O}(2p)) = 2$ implies that a nonconstant section $f \in H^0(\tilde{C}, \mathcal{O}(2p))$ satisfies $f(a_i) = f(b_i)$ for $i = 1, \dots, s$. Pick an element $h \in H^0(\tilde{C}, \mathcal{O}(3p)) \setminus H^0(\tilde{C}, \mathcal{O}(2p))$. Since the sections $(1, f, h)$ form a basis of $H^0(\tilde{C}, \mathcal{O}(3p))$, they distinguish points of \tilde{C} , so we have $h(a_i) \neq h(b_i)$, and h cannot descend to an element of $H^0(C, \mathcal{O}(3p))$. Hence, $H^0(C, \mathcal{O}(3p)) = H^0(C, \mathcal{O}(2p))$ in this case.

Finally, consider the case $\tilde{C} = \mathbb{P}^1$. We can assume that $p = \infty$ and think of sections of $\mathcal{O}(np)$ on \mathbb{P}^1 as polynomials of degree n . Without loss of generality we can assume that $b_i = -a_i$ for all i (so that $t^2 \in H^0(\mathbb{P}^1, \mathcal{O}(2p))$ descends to a non-constant section of $\mathcal{O}(2p)$ on C). Assume that there is a polynomial h of degree 3 such that $h(a_i) = h(-a_i)$ for all i . Write $h = h_+ + h_-$, where h_+ is even and h_- is odd. Then we have $h_-(a_i) = 0$ for $i = 1, \dots, s$. Since h_- is an odd cubic polynomial, this implies that $s = 1$, which contradicts to the assumption $g \geq 2$. \square

Remark 2.5.4. There are examples of (C, p) , such that $H^0(C, \mathcal{O}(3p)) \not\subset H^0(C, \mathcal{O}(2p)) \not\subset H^0(C, \mathcal{O}(p))$ and the arithmetic genus of C is ≥ 2 (necessarily with C reducible). The simplest example of genus 2 is the union of two elliptic curves intersecting at one point (p can be any smooth point).

In the case when C is a hyperelliptic smooth projective curve we can calculate the Massey product (2.5.1) in terms of the corresponding ramification points on \mathbb{P}^1 . Let $f : C \rightarrow \mathbb{P}^1$ be the morphism given by the hyperelliptic linear system, so that $\mathcal{O}(2p) \simeq f^*\mathcal{O}(1)$. Let p_1, \dots, p_g be distinct Weierstrass points on C (i.e., ramification points of f), and let $a_i = f(p_i) \in \mathbb{P}^1$. Then setting $D_i = \sum_{j \neq i} p_j$ we can use an isomorphism $H^0(\mathcal{O}(D_i)/\mathcal{O}) \simeq H^1(\mathcal{O})/\langle \xi_i \rangle$ (where $\xi_i = \xi_{p_i}$) and view the Massey product (2.5.1) for $p = p_i$ as a map

$$T_{p_i}^{\otimes 3} \rightarrow H^0(\mathcal{O}(D_i)/\mathcal{O}) \simeq \bigoplus_{j \neq i} T_{p_j}.$$

Let $\alpha_{ij}^{he} : T_{p_i}^{\otimes 3} \rightarrow T_{p_j}$ be the components of this map, where $i \neq j$.

Proposition 2.5.5. *Let $f : C \rightarrow \mathbb{P}^1$ be a hyperelliptic covering associated with a separable form F of degree $2g + 2$, and let p_1, \dots, p_g be distinct ramification points with $f(p_i) = a_i \in \mathbb{P}^1$.*

(a) *Set $T_{a_i} = T_{a_i}\mathbb{P}^1$. There is a natural isomorphism*

$$T_{p_j} \otimes T_{p_i}^{-3} \xrightarrow{\kappa_{ij}} \mathcal{O}(-g+1)_{a_j} \otimes \mathcal{O}(g-1)_{a_i} \otimes T_{a_i}^{-1} \quad (2.5.4)$$

such that $\kappa_{ij}(\alpha_{ij}^{he})$ depends only on the g -tuple $a_1, \dots, a_g \in \mathbb{P}^1$ (and not on the form F).

(b) *Assume that $a_i \in \mathbb{P}^1 \setminus \{\infty\}$ for $i = 1, \dots, g$. Then using the natural trivialization of the right-hand side of (2.5.4) we have*

$$\kappa_{ij}(\alpha_{ij}^{he}) = \frac{1}{a_j - a_i} \cdot \prod_{k \neq i, j} \frac{a_i - a_k}{a_j - a_k}.$$

Proof. (a) Let $p = p_i$. Let $\xi_p \in H^1(C, \mathcal{O}) \otimes T_p^{-1}$ be the image of the canonical generator of $H^0(C, \mathcal{O}(p)/\mathcal{O}) \otimes T_p^{-1}$ under the boundary homomorphism. Recall that the Massey product (2.5.1) is given by the $\langle \xi_p \rangle$ -coset $s_*(t_*)^{-1}(\xi_p)$ determined from the diagram

$$H^1(C, \mathcal{O}) \otimes T_p^{-1} \xleftarrow{t_*} H^1(C, \mathcal{O}(-2p)) \otimes T_p^{-1} \xrightarrow{s_*} H^1(C, \mathcal{O}) \otimes T_p^{-3},$$

where $t = 1 \in H^0(C, \mathcal{O}(2p))$ and $s \in H^0(C, \mathcal{O}(2p)) \otimes T_p^{-2}$ is any representative in the $H^0(C, \mathcal{O}(p)) \otimes T_p^{-2}$ -coset projecting to the natural generator of $\mathcal{O}(2p)/\mathcal{O}(p) \otimes T_p^{-2}$. By

Serre duality, we can rewrite the above diagram as

$$H^0(C, \omega_C)^* \otimes T_p^{-1} \xleftarrow{t^*} H^0(C, \omega_C(2p))^* \otimes T_p^{-1} \xrightarrow{s^*} H^0(C, \omega_C)^* \otimes T_p^{-3}, \quad (2.5.5)$$

We can realize C as the relative spectrum of the sheaf of $\mathcal{O}_{\mathbb{P}^1}$ -algebras

$$\mathcal{A} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-g-1),$$

where the product $\mathcal{O}_{\mathbb{P}^1}(-g-1) \otimes \mathcal{O}_{\mathbb{P}^1}(-g-1) \rightarrow \mathcal{O}$ is given by the form $F \in H^0(\mathbb{P}^1, \mathcal{O}(2g+2))$, vanishing in $2g+2$ ramification points of f . By the relative duality, we have

$$f_*\omega_C \simeq \omega_{\mathbb{P}^1} \otimes \mathcal{A}^\vee \simeq \det(V) \otimes \mathcal{A}(g-1), \quad (2.5.6)$$

where we use a canonical isomorphism $\omega_{\mathbb{P}^1} \simeq \det(V) \otimes \mathcal{O}_{\mathbb{P}^1}(-2)$ for $\mathbb{P}^1 = \mathbb{P}(V)$ and the isomorphism of \mathcal{A} -modules $\mathcal{A}^\vee \simeq \mathcal{A}(g+1)$. This induces an isomorphism

$$T_p^{-1} = \omega_C|_p \simeq \det(V) \otimes \mathcal{O}(g-1)_a, \quad (2.5.7)$$

Together with a natural isomorphism $T_a^{-1} \simeq T_p^{-2}$ this immediately leads to (2.5.4).

On the other hand, we have

$$f_*\omega_C(2p) \simeq \det(V) \otimes \mathcal{A}(g-1)(a).$$

Therefore, the diagram (2.5.5) is isomorphic to the twist by $\det(V)^{-1} \otimes T_p^{-1}$ of the diagram

$$H^0(\mathbb{P}^1, \mathcal{O}(g-1))^* \xleftarrow{t^*} H^0(\mathbb{P}^1, \mathcal{O}(g-1)(a))^* \xrightarrow{s^*} H^0(\mathbb{P}^1, \mathcal{O}(g-1))^* \otimes T_a^{-1},$$

where t^* is dual to the natural embedding $H^0(\mathbb{P}^1, \mathcal{O}(g-1)) \hookrightarrow H^0(\mathbb{P}^1, \mathcal{O}(g-1)(a))$, and s^* is induced by a global section s of $\mathcal{O}_{\mathbb{P}^1}(a) \otimes T_a^{-1}$ such that $s(a) = 1$. The element $\xi_p \in H^1(C, \mathcal{O})$ corresponds to the evaluation functional $H^0(C, \omega_C) \rightarrow \omega_C|_p = T_p^{-1}$. Thus, under the isomorphism (2.5.7) ξ_p corresponds to the natural evaluation functional $\text{ev}_a \in H^0(\mathbb{P}^1, \mathcal{O}(g-1))^* \otimes \mathcal{O}(g-1)_a$. Recall that we need the above picture for $p = p_i$, so below we will write $s = s_i$, and $\text{ev}_i = \text{ev}_{a_i}$ (where $a_i = f(p_i)$). One more ingredient in the construction of α_{ij}^{he} is the direct sum decomposition

$$H^1(C, \mathcal{O}_C) \simeq \bigoplus_{j=1}^g T_{p_j}$$

induced by the elements $\xi_{p_j} \in T_{p_j} \otimes H^1(C, \mathcal{O})$. In terms of the above isomorphism the summands of this decomposition correspond to the lines $\langle \text{ev}_{a_j} \rangle \in H^0(\mathbb{P}^1, \mathcal{O}(g-1))^*$. Thus, the projection $H^1(C, \mathcal{O}_C) \rightarrow T_{p_j}$ is dual (up to the twist by $\det(V)$) to the embedding

$$\mathcal{O}(g-1)_{a_j} \xrightarrow{f_j} H^0(\mathbb{P}^1, \mathcal{O}(g-1)),$$

where the global section $f_j \in H^0(\mathbb{P}^1, \mathcal{O}(g-1)) \otimes \mathcal{O}(-g+1)_{a_j}$ is characterized by $f_j(a_k) = 0$ for $k \neq j$ and $f_j(a_j) = 1$. Note that for $i \neq j$ we have $f_j(a_i) = 0$, so the product $s_i f_j$ is a well-defined element of $H^0(\mathbb{P}^1, \mathcal{O}(g-1)) \otimes \mathcal{O}(-g+1)_{a_j} \otimes T_{a_i}^{-1}$. Thus,

$$\kappa_{ij}(\alpha_{ij}^{he}) = (s_i f_j)(a_i) \in \mathcal{O}(-g+1)_{a_j} \otimes \mathcal{O}(g-1)_{a_i} \otimes T_{a_i}^{-1}, \quad (2.5.8)$$

This implies our assertion.

(b) Now assuming that all $a_j \in \mathbb{P}^1 \setminus \{\infty\}$ we can use the trivializations of $\mathcal{O}(1)_{a_j}$ induced by the section $x_0 \in \mathcal{O}(1)$, where $(x_0 : x_1)$ are homogeneous coordinates on \mathbb{P}^1 , and the trivialization of T_{a_i} given by $\frac{d}{dt}$, where $t = x_1/x_0$. Then we can take $s_i = \frac{1}{t-a_i}$ and

$$f_j = \prod_{k \neq j} \frac{t - a_k}{a_j - a_k} \cdot x_0^{g-1}.$$

It remains to use (2.5.8). □

2.6. Consequences for the A_∞ -infinity structure. Let (C, p_1, \dots, p_n) be a smooth projective curve of genus $g \geq 1$ with n marked points (where $n \geq 1$), such that $h^0(p_1 + \dots + p_n) = 1$. Let $E = E_{g,n}$ be the Ext-algebra of the generator $\mathcal{O}_C \oplus \mathcal{O}_{p_1} \oplus \dots \oplus \mathcal{O}_{p_n}$ of $D^b(C)$. By the homological perturbation theory, we have a minimal A_∞ -structure on E extending the associative product on E , defined uniquely up to A_∞ -equivalence.

Theorem 2.6.1. *The A_∞ -structure on E coming from the data (C, p_1, \dots, p_n) is equivalent to the one with $m_3 = 0$ if and only if either $g = 1$ or C is hyperelliptic and p_1, \dots, p_n are Weierstrass points. If $m_3 = 0$ and $g > 1$ then m_4 is always nontrivial.*

Proof. Assume first that $n = g$. A minimal A_∞ -structure is equivalent to the one with $m_3 = 0$ if and only if the Hochschild cohomology class given by m_3 is trivial. By Proposition 1.3.3, this happens exactly when

$$m_3(\eta_i, \psi_i, \theta_i) \in \langle \xi_i \rangle$$

for all $i = 1, \dots, n$. By Corollary 2.3.2, this is equivalent to the vanishing of the Massey products (2.4.1) for $p = p_1, \dots, p_n$. Now Proposition 2.4.1 tells that this is equivalent to C being hyperelliptic and p_1, \dots, p_n being Weierstrass points.

In the case $n < g$ considering the same Massey products shows that the condition for C to be hyperelliptic and for p_1, \dots, p_n to be Weierstrass points is necessary. Conversely, if we have such n -tuple of Weierstrass points on a hyperelliptic curve we can complete it to a g -tuple of Weierstrass points p_1, \dots, p_g still satisfying the condition $h^0(p_1 + \dots + p_g) = 1$ (see Lemma 2.6.2 below). By the first part of the proof, the A_∞ -structure on $\mathcal{O}, \mathcal{O}_{p_1}, \dots, \mathcal{O}_{p_g}$ can be chosen to have trivial m_3 , as required.

The second assertion follows from the nontriviality of the quadruple Massey product (2.5.1) for a Weierstrass point on a hyperelliptic curve (see Corollary 2.5.3). To connect this Massey product to m_4 we use Corollary 2.3.2 noting that the needed dg Massey product is defined by Lemma 2.5.1. □

Lemma 2.6.2. *Let p_1, \dots, p_n be distinct Weierstrass points on a hyperelliptic curve, where $n \leq g$. Then $h^0(p_1 + \dots + p_n) = 1$.*

Proof. It is enough to consider the case $n = g$ in which case we have to check that $h^1(D) = h^0(K - D) = 0$, where $D = p_1 + \dots + p_g$. Indeed, otherwise we would have $K = D + D'$ for some effective divisor D' of degree $g - 2$. Since every effective canonical divisor on C is a sum of $g - 1$ fibers of the hyperelliptic map $f : C \rightarrow \mathbb{P}^1$, this would imply that $f(D + D')$ is supported at $\leq g - 1$ points, which is a contradiction. □

3. RATIONAL FUNCTIONS ON $\mathcal{M}_{g,g}$ ASSOCIATED WITH MASSEY PRODUCTS

3.1. Triple products as sections of line bundles over the moduli spaces. Let C be a projective curve of arithmetic genus $g \geq 2$, and let p_1, \dots, p_g be distinct smooth points such that $h^0(p_1 + \dots + p_g) = 1$. Then by Proposition 1.3.3 and Corollary 2.4.3, the Hochschild class of m_3 on $E_{g,g}$ (where the A_∞ -structure comes from (C, p_1, \dots, p_g)) is determined by the collection of elements $\alpha_{ij} \in \text{Hom}_k(T_{p_i}^{\otimes 2}, T_{p_j})$, $i \neq j$, given by

$$\alpha_{ij} = r_{D_i+2p_i, p_j} \circ r_{D_i+2p_i, p_i}^{-1}, \quad (3.1.1)$$

where we use the restriction maps

$$\begin{aligned} r_{D_i+2p_i, p_i} : H^0(\mathcal{O}(D_i + 2p_i))/H^0(\mathcal{O}) &\xrightarrow{\sim} H^0(\mathcal{O}(2p_i)/\mathcal{O}(p_i)) \simeq T_{p_i}^{\otimes 2} \\ r_{D_i+2p_i, p_j} : H^0(\mathcal{O}(D_i + 2p_i))/H^0(\mathcal{O}) &\rightarrow H^0(\mathcal{O}(p_j)/\mathcal{O}) \simeq T_{p_j}, \end{aligned} \quad (3.1.2)$$

where $D_i = \sum_{j \neq i} p_j$. In particular, this construction makes sense over the open substack $\mathcal{U} \subset \overline{\mathcal{M}}_{g,g}$ of the Deligne-Mumford stack of stable curves with marked points, corresponding to (C, p_1, \dots, p_g) such that $h^0(p_1 + \dots + p_g) = 1$. Thus, α_{ij} can be viewed as a section over \mathcal{U} of the line bundle $L_i^2 \otimes L_j^{-1}$, where $L_i := p_i^* K$ (the pullback of the relative canonical class on the universal curve).

Let us set $D_{ij} = \sum_{m \neq i,j} p_m$. The zero locus of α_{ij} is supported on the divisor $Z_{ij} \subset \overline{\mathcal{M}}_{g,g}$ of (C, p_1, \dots, p_g) such that $h^0(2p_i + D_{ij}) > 1$. In particular, α_{ij} is nonzero. The complement to \mathcal{U} is the divisor $Z \subset \overline{\mathcal{M}}_{g,g}$ of (C, p_1, \dots, p_g) such that $h^0(p_1 + \dots + p_g) > 1$. More precisely, we define Z as the degeneration locus of the map $H^0(\mathcal{O}(p_1 + \dots + p_g)/\mathcal{O}) \rightarrow H^1(\mathcal{O})$, which is the zero locus of a section of the line bundle $\det(\Lambda)^{-1} \otimes L_1 \otimes \dots \otimes L_g$ on $\overline{\mathcal{M}}_{g,g}$, where Λ is the Hodge bundle. Similarly, Z_{ij} is defined as the degeneration locus of the map $H^0(\mathcal{O}(2p_i + D_{ij})/\mathcal{O}) \rightarrow H^1(\mathcal{O})$, so it is the zero locus of a section of $\det(\Lambda)^{-1} \otimes L_i^3 \otimes \bigotimes_{m \neq i,j} L_m$.

Note that the divisors Z and Z_{ij} have in general many irreducible components (they contain some boundary components).

Proposition 3.1.1. *The section $\alpha_{ij} \in \Gamma(\mathcal{U}, L_i^2 \otimes L_j^{-1})$ extends to a global section*

$$\tilde{\alpha}_{ij} \in \Gamma(\overline{\mathcal{M}}_{g,g}, L_i^2 \otimes L_j^{-1}(Z))$$

such that the zero locus of $\tilde{\alpha}_{ij}$ is exactly Z_{ij} .

The proof will be based on the following general fact from tensor algebra. Recall that for a morphism of vector bundles $\phi : V \rightarrow W$ such that $r = \text{rk } W = \text{rk } V - 1$ one has a canonical map

$$k_\phi : \det(V) \otimes \det(W^*) \rightarrow V$$

such that $\phi \circ k_\phi = 0$. Namely, k_ϕ is obtained by tensoring with $\det(V)$ from the map

$$\bigwedge^r(\phi^*) : \det(W^*) \rightarrow \bigwedge^r(V^*)$$

using the natural isomorphism $\det(V) \otimes \bigwedge^r(V^*) \simeq V$.

Lemma 3.1.2. *Let $0 \rightarrow V_1 \xrightarrow{\iota} V \xrightarrow{\pi} L \rightarrow 0$ be an exact sequence of vector bundles, where L is a line bundle, and let $\phi : V \rightarrow W$ be a morphism of vector bundles, where*

$r = \text{rk } W = \text{rk } V_1 = \text{rk } V - 1$. Let $Z \subset S$ be the degeneration divisor of the restriction $\phi_1 = \phi|_{V_1} : V_1 \rightarrow W$. Then Z coincides with the vanishing locus of the composed map

$$\det(V) \otimes \det(W^*) \xrightarrow{k_\phi} V \xrightarrow{\pi} L.$$

Proof. Note that Z is the vanishing locus of $\det(\phi_1) : \det(V_1) \rightarrow \det(W)$, or equivalently, of the dual map $\det(\phi_1^*)$. Thus, the assertion follows from the commutativity of the diagram

$$\begin{array}{ccccc} \det(V) \otimes \det(W^*) & \xrightarrow{\bigwedge^r(\phi^*)} & \det(V) \otimes \bigwedge^r(V^*) & \xrightarrow{\sim} & V \\ & \searrow \det(\phi_1^*) & \downarrow \bigwedge^r(\iota^*) & & \downarrow \pm\pi \\ & & \det(V) \otimes \det(V_1^*) & \xrightarrow{\sim} & L \end{array}$$

since the composition of arrows in the top row is k_ϕ . \square

Proof of Proposition 3.1.1. Let V be the bundle on $\overline{\mathcal{M}}_{g,g}$ with the fiber $H^0(C, \mathcal{O}(2p_i + D_i)/\mathcal{O})$ over (C, p_1, \dots, p_g) , and let $W = \Lambda^*$, so the fiber of W at (C, p_1, \dots, p_g) is $H^1(C, \mathcal{O})$. We have a natural connecting homomorphism $\phi : V \rightarrow W$. We have natural restriction maps $\pi_i : V \rightarrow L_i^{-2}$ and $\pi_j : V \rightarrow L_j^{-1}$. Applying Lemma 3.1.2 to the exact sequence of bundles

$$0 \rightarrow V' \rightarrow V \rightarrow L_i^{-2} \rightarrow 0,$$

where V' is the bundle on $\overline{\mathcal{M}}_{g,g}$ with the fiber $H^0(C, \mathcal{O}(p_i + D_i))$ we see that the divisor $Z \subset \overline{\mathcal{M}}_{g,g}$ coincides with the vanishing locus of the composition

$$\det(V) \otimes \det(W^*) \xrightarrow{k_\phi} V \xrightarrow{\pi_i} L_i^{-2}.$$

Note that over \mathcal{U} the image of k_ϕ generates $\ker(\phi)$, and $\ker(\phi)$ is a bundle with the fiber

$$\ker(H^0(C, \mathcal{O}(2p_i + D_i)/\mathcal{O}) \rightarrow H^1(C, \mathcal{O})) \simeq H^0(C, \mathcal{O}(2p_i + D_i))/H^0(C, \mathcal{O}).$$

Thus, we can replace the restriction maps $r_{D_i+2p_i, p_i}$ and $r_{D_i+2p_i, p_j}$ used in defining α_{ij} (see (3.1.2)) with the morphisms $\pi_i \circ k_\phi$ and $\pi_j \circ k_\phi$, respectively. Since $\pi_i \circ k_\phi$ induces an isomorphism

$$\det(V) \otimes \det(W^*) \simeq L_i^{-2}(-Z),$$

we obtain the global morphism

$$L_i^{-2}(-Z) \simeq \det(V) \otimes \det(W^*) \xrightarrow{\pi_j \circ k_\phi} L_j^{-1}$$

which gives the required global section $\tilde{\alpha}_{ij} \in \Gamma(\overline{\mathcal{M}}_{g,g}, L_i^2 \otimes L_j^{-1}(Z))$. Now applying Lemma 3.1.2 to the exact sequence

$$0 \rightarrow V'' \rightarrow V \rightarrow L_j^{-1} \rightarrow 0,$$

where V'' is the bundle on $\overline{\mathcal{M}}_{g,g}$ with the fiber $H^0(C, \mathcal{O}(2p_i + D_{ij}))$, we see that the vanishing locus of $\pi_j \circ k_\phi$ is exactly Z_{ij} . \square

Remark 3.1.3. It is not essential to work with stable curves in the above argument. The result similar to Proposition 3.1.1 would work with other modular compactifications of $\mathcal{M}_{g,g}$.

3.2. Rational functions. Let $\mathcal{M}_{g,g}^{(1)} \rightarrow \mathcal{M}_{g,g}$ (resp., $\overline{\mathcal{M}}_{g,g}^{(1)} \rightarrow \overline{\mathcal{M}}_{g,g}$) be the \mathbb{G}_m^g -torsor corresponding to choices of nonzero tangent vectors at each of the marked points. Then the line bundles L_i are naturally trivialized on $\mathcal{M}_{g,g}^{(1)}$, so we can view each section α_{ij} as a rational function on $\mathcal{M}_{g,g}^{(1)}$. This gives a rational map

$$\alpha : \mathcal{M}_{g,g}^{(1)} \xrightarrow{(\alpha_{ij})} \mathbb{G}_m^{g^2-g}. \quad (3.2.1)$$

On the other hand, considering rational monomials in α_{ij} we can get rational functions on $\mathcal{M}_{g,g}$. Namely, consider the homomorphism of groups

$$\varphi : \mathbb{Z}^{g^2-g} \rightarrow \mathbb{Z}^g : e_{ij} \rightarrow 2e_i - e_j,$$

where \mathbb{Z}^{g^2-g} (resp., \mathbb{Z}^g) has a basis $(e_{ij})_{i \neq j}$ (resp., e_i), where $i, j \leq g$. Then for every element $x = \sum n_{ij} e_{ij} \in \ker(\varphi)$ the expression

$$\alpha^x := \prod \alpha_{ij}^{n_{ij}}$$

is a rational function on $\mathcal{M}_{g,g}$. It is easy to see that $\ker(\varphi)$ has rank $g^2 - 2g$, so choosing a basis b_1, \dots, b_{g^2-2g} in $\ker(\varphi)$ we obtain a rational map

$$\overline{\alpha} : \mathcal{M}_{g,g} \xrightarrow{\alpha^{b_1}, \dots, \alpha^{b_{g^2-2g}}} \mathbb{G}_m^{g^2-2g}. \quad (3.2.2)$$

Note the rational map (3.2.1) is \mathbb{G}_m^g -equivariant, where $(\lambda_1, \dots, \lambda_g)$ acts on $\mathbb{G}_m^{g^2-g}$ via the homomorphism $\varphi^* : \mathbb{G}_m^g \rightarrow \mathbb{G}_m^{g^2-g}$, dual to φ , and the map $\overline{\alpha}$ can be viewed as the induced rational map of quotients by \mathbb{G}_m^g .

Theorem 3.2.1. *Let $\text{char}(\mathbb{k}) = 0$. If $g \geq 6$ then the map (3.2.2) is birational onto its image.*

The proof of this theorem will be given in Section 4. The result is optimal, since for $g \leq 5$ we have $\dim \mathcal{M}_{g,g} > g^2 - 2g$. In fact, for $g \leq 5$ the map (3.2.2) is dominant (see Theorem 5.2.2 below).

Proposition 3.2.2. *Let $g \geq 3$. For a generic curve C the restriction of $\overline{\alpha}$ gives a rational map*

$$\overline{\alpha}_C : C^g \rightarrow \mathbb{G}_m^{g^2-2g}$$

with generically injective tangent map. Hence, the image of $\overline{\alpha}_C$ has dimension g .

Proof. Using the sections α_{ij} on $\mathcal{U} \subset \overline{\mathcal{M}}_{g,g}$ (see Section 3.1) we can extend the map $\overline{\alpha}$ to stable curves. It is enough to construct a stable curve (C, p_1, \dots, p_g) in \mathcal{U} for which the assertion is true. Let us consider the wheel of \mathbb{P}^1 's with g components C_1, \dots, C_g , so that $1 \in C_i$ is glued to $0 \in C_{i+1}$ (we think of indices as elements of $\mathbb{Z}/g\mathbb{Z}$). Now consider the nodal curve C obtained as the union of this wheel with one more component $C_\infty \simeq \mathbb{P}^1$ which intersects each component C_i at one point $\infty \in C_i$ (we fix the corresponding g distinct points on C_∞). Note that the arithmetic genus of C is g . We choose marked points p_1, \dots, p_g , so that $p_i = \lambda_i \in C_i \setminus \{0, 1, \infty\}$.

Let us compute α_{1i} . By definition, for this we have to produce a non-constant element $f \in H^0(C, \mathcal{O}(2p_1 + p_2 + \dots + p_g))$. Such a function is given by a collection of functions $(f_1, \dots, f_g, f_\infty)$, where $f_1 \in H^0(C_1, \mathcal{O}(2p_1))$, $f_i \in H^0(C_i, \mathcal{O}(2p_i))$ for $i = 2, \dots, g$ and f_∞ is a constant, subject to the constraints

$$f_i(1) = f_{i+1}(0),$$

$$f_i(\infty) = f_\infty,$$

where $i=1, \dots, g$. Subtracting a constant from f we can assume that $f_\infty = 0$. Then we can take

$$f_1(t) = \frac{1}{(t - \lambda_1)^2} + \frac{y_1}{t - \lambda_1},$$

$$f_i(t) = \frac{y_i}{t - \lambda_i},$$

for some constants y_1, \dots, y_g , and the equations become

$$\frac{1}{(1 - \lambda_1)^2} + \frac{y_1}{1 - \lambda_1} = -\frac{y_2}{\lambda_2},$$

$$\frac{y_i}{1 - \lambda_i} = -\frac{y_{i+1}}{\lambda_{i+1}}, \quad i = 2, \dots, g-1,$$

$$\frac{y_g}{1 - \lambda_g} = \frac{1}{\lambda_1^2} - \frac{y_1}{\lambda_1}.$$

Solving this system we obtain

$$\alpha_{12} = y_2 = \frac{\lambda_2}{\lambda_1(\lambda_1 - 1)^2(a - 1)}, \quad \alpha_{13} = y_3 = \frac{\lambda_2\lambda_3}{\lambda_1(\lambda_1 - 1)^2(\lambda_2 - 1)(a - 1)}, \quad \text{etc.},$$

where

$$a = \frac{\lambda_1\lambda_2 \dots \lambda_g}{(\lambda_1 - 1)(\lambda_2 - 1) \dots (\lambda_g - 1)}.$$

Now we find

$$\frac{\alpha_{i,i+1}^2 \alpha_{i+1,i+3}}{\alpha_{i,i+2}^2 \alpha_{i+2,i+3}} = \frac{\lambda_{i+2} - 1}{\lambda_{i+1}} \quad \text{for } i = 1, \dots, g,$$

which implies that the parameters $\lambda_1, \dots, \lambda_g$ can be recovered from the image of the map (3.2.2). \square

Example 3.2.3. In the case $g = 2$ the homomorphism $\varphi^* : \mathbb{G}_m^2 \rightarrow \mathbb{G}_m^2$ has kernel $\mathbb{Z}/3\mathbb{Z} \subset \mathbb{G}_m^2$, generated by (ζ_3, ζ_3^{-1}) , where ζ_3 is a primitive 3rd root of unity. Hence, the map α in this case factors through a rational map

$$\alpha' : \mathcal{M}_{2,2}^{(1)}/(\mathbb{Z}/3\mathbb{Z}) \rightarrow \mathbb{G}_m^2.$$

The generic fibers of this map $(\alpha')^{-1}(\lambda, \mu)$ are rational sections for the projection

$$\mathcal{M}_{2,2}^{(1)}/(\mathbb{Z}/3\mathbb{Z}) \rightarrow \mathcal{M}_{2,2}.$$

More explicitly, for (C, p_1, p_2) there is a unique choice of nonzero tangent vectors $(v_1 \in T_{p_1}, v_2 \in T_{p_2})$, up to the $\mathbb{Z}/3\mathbb{Z}$ -action generated by $(v_1, v_2) \mapsto (\zeta_3 v_1, \zeta_3^{-1} v_2)$. Namely, v_1 and

v_2 are defined by the condition that there exist rational functions $f_1 \in H^0(C, \mathcal{O}(2p_1 + p_2))$, $f_2 \in H^0(C, \mathcal{O}(p_1 + 2p_2))$ with

$$\begin{aligned} f_1 &\equiv v_1^2 \bmod \mathcal{O}(p_1 + p_2), \quad f_1 \equiv \lambda v_2 \bmod \mathcal{O}(2p_1), \\ f_2 &\equiv v_2^2 \bmod \mathcal{O}(p_1 + p_2), \quad f_2 \equiv \mu v_1 \bmod \mathcal{O}(2p_2). \end{aligned}$$

Example 3.2.4. In the case $g = 3$ the space $\mathcal{M}_{3,3}^{(1)}$ is 12-dimensional. By Proposition 3.2.2, for generic curve C of genus 3 the rational map

$$\bar{\alpha}_C : C^3 \rightarrow \mathbb{G}_m^3$$

is generically étale. Hence, at generic point of $\mathcal{M}_{3,3}^{(1)}$ the fibers of the two dominant (rational) maps to 6-dimensional spaces

$$\begin{array}{ccc} \mathcal{M}_{3,3}^{(1)} & \xrightarrow{\alpha} & \mathbb{G}_m^6 \\ \pi \downarrow & & \\ \mathcal{M}_3 & & \end{array}$$

are transversal.

3.3. Interpretation in terms of tangent lines. Let C be a smooth projective curve of genus $g \geq 2$. Let L be a base point free line bundle on C . For a point $p \in C$ let $\text{ev}_p \in L|_p \otimes H^0(C, L)^*$ denote the functional of evaluation at p . Then the tangent map at a point $p \in C$ to the map

$$\varphi_L : C \xrightarrow{|L|} \mathbb{P}(H^0(C, L)^*),$$

given by the linear system $|L|$, is the map

$$T_p C \rightarrow L|_p \otimes H^0(C, L)^* / \langle \text{ev}_p \rangle \simeq L|_p \otimes H^0(C, L(-p))^*,$$

dual to the evaluation functional for $L(-p)$,

$$H^0(C, L(-p)) \rightarrow L(-p)|_p \simeq (T_p C)^* \otimes L|_p.$$

In the case of the canonical line bundle $L = \omega_C$, under the duality $H^0(C, \omega_C)^* \simeq H^1(C, \mathcal{O}_C)$ the functional ev_p corresponds to the element $\xi_p \in H^1(C, \mathcal{O}_C)$, obtained from the connecting homomorphism $H^0(\mathcal{O}(p)/\mathcal{O}) \rightarrow H^1(\mathcal{O})$. Hence, the tangent map to the canonical morphism $\varphi_{\omega_C} : C \rightarrow \mathbb{P}(H^0(C, \omega_C)^*)$ at $p \in C$ can be identified with the connecting homomorphism

$$\delta'_p : T_p C \simeq T_p^* C \otimes H^0(C, \mathcal{O}(2p)/\mathcal{O}(p)) \rightarrow T_p^* C \otimes H^1(C, \mathcal{O}(p)) \simeq T_p^* C \otimes H^1(C, \mathcal{O}) / \langle \xi_p \rangle,$$

which is exactly the triple Massey product considered in Section 2.4.

Now recall that for g distinct points $p_1, \dots, p_g \in C$ such that $h^0(p_1 + \dots + p_g) = 1$ the maps $\alpha_{ij} : T_{p_i}^2 \rightarrow T_{p_j}$ for $i \neq j$, considered above, can be identified with the components of the same Massey product for $p = p_i$, up to a sign (see Proposition 2.4.2 and Corollary 2.4.3). This leads to the following identification of the rows of the matrix $(-\alpha_{ij})$ with the coordinates of the tangent map to the morphism given by $|\omega_C|$.

Proposition 3.3.1. *The components of the tangent map δ'_{p_i} to φ_{ω_C} at p_i , with respect to the decomposition $H^1(C, \mathcal{O}) \simeq \bigoplus_j T_{p_j}$, are given by tensoring with $-\alpha_{ij}$.*

Note that the position of the tangent line to C at p_i in \mathbb{P}^{g-1} is recorded by (α_{ij}) with fixed i , viewed as homogeneous coordinates. In order to recover the same data as the map $\bar{\alpha}$, we note that there is a canonical identification of the tangent line to C at p_i with the fiber of the tautological line bundle $\mathcal{O}_{\mathbb{P}^{g-1}}(-1)$ at p_i . Thus, Theorem 3.2.1 leads to the following result.

Corollary 3.3.2. *Let $\text{char}(\mathbb{k}) = 0$ and $g \geq 6$. Let us associate with generic $(C, p_\bullet) \in \mathcal{M}_{g,g}$ the collection (for $i = 1, \dots, g$) of points $x_i = \varphi_{\omega_C}(p_i) \in \mathbb{P}^{g-1}$ and of tangent lines $L_i \subset \mathbb{P}^{g-1}$ to $\varphi_{\omega_C}(C)$ at x_i together with identification of each tangent space $T_{x_i}L_i$ with the fiber of the tautological line bundle $\mathcal{O}(-1)|_{x_i}$. Then generic (C, p_\bullet) can be recovered from these data (viewed up to projective transformations).*

Next, consider the map $C \rightarrow \mathbb{P}^g$ given by the linear system $|2D|$, where $D = \sum_{i=1}^g p_i$. If $h^0(2D - K) = 0$ (which is true generically) then this map is an embedding and its image is a degree $2g$ curve in \mathbb{P}^g . Note that the section $1 \in H^0(C, \mathcal{O}(2D))$ corresponds to a hyperplane $H \subset \mathbb{P}^g$ which is tangent to C at all g points p_1, \dots, p_g . Also the condition $h^0(D) = 1$ means that p_1, \dots, p_g are in general position in H .

Now suppose we are given any degree $2g$ curve $C \subset \mathbb{P}^g$, and a linear form $\ell \in H^0(\mathbb{P}^g, \mathcal{O}(1))$ such that the corresponding hyperplane $H = (\ell = 0)$ is tangent to C at g points p_1, \dots, p_g that are smooth points of C and are in general linear position (we assume also that $C \not\subset H$). Let $L = \mathcal{O}(1)|_C$. Since $\deg(L) = 2g$ and the section ℓ vanishes along the divisor $2p_1 + \dots + 2p_g$, it induces an isomorphism

$$\mathcal{O}(1)|_{p_i} \simeq \mathcal{O}_C(2p_i)/\mathcal{O}(p_i) \simeq T_{p_i}^2 \quad (3.3.1)$$

for each i . Since p_1, \dots, p_g are in general position, we obtain an isomorphism

$$H^0(H, \mathcal{O}(1)) \simeq \bigoplus_{i=1}^g \mathcal{O}(1)|_{p_i} \simeq \bigoplus_{i=1}^g T_{p_i}^2.$$

Therefore, we have a canonical isomorphism

$$T_{p_i}H \simeq \bigoplus_{j \neq i} T_{p_i}^2 \otimes T_{p_j}^{-2}. \quad (3.3.2)$$

Now, since H is tangent to C at each p_i , the tangent map to the embedding $C \rightarrow \mathbb{P}^g$ at p_i gives a linear map

$$T_{p_i} \rightarrow T_{p_i}H \simeq \bigoplus_{j \neq i} T_{p_i}^2 \otimes T_{p_j}^{-2}. \quad (3.3.3)$$

This time the tangent map will be given by a column of the matrix (α_{ij}) .

Proposition 3.3.3. *Suppose $C \hookrightarrow \mathbb{P}^g$ is a smooth projective curve embedded by the linear system $|2D|$, where $D = \sum_{i=1}^g p_i$ and $h^0(D) = 1$. Also, let ℓ be the section $1 \in H^0(C, \mathcal{O}(2D)) \simeq H^0(\mathbb{P}^g, \mathcal{O}(1))$. Then the components of the map (3.3.3) are given by tensoring with α_{ji} (see (3.1.1)).*

Proof. We have $L = \mathcal{O}(1)|_C = \mathcal{O}_C(2D)$. The point p_i corresponds to the functional

$$H^0(\mathbb{P}^g, \mathcal{O}(1)) = H^0(C, L) \rightarrow L|_{p_i},$$

and the tangent map to the embedding $C \rightarrow \mathbb{P}^g$ at p_i corresponds to the dual of the natural restriction map

$$H^0(C, L(-p_i))/(1) \rightarrow L(-p_i)|_{p_i} \simeq L|_{p_i} \otimes (T_{p_i}C)^*. \quad (3.3.4)$$

The components of the direct sum decomposition (3.3.2) are exactly the subspaces

$$H^0(C, \mathcal{O}(2p_i + D_i))/(1) \subset H^0(C, L)/(1) \simeq H^0(H, \mathcal{O}(1)).$$

The subspace $H^0(C, L(-p_i))/(1) \subset H^0(C, L)/(1)$ is the direct sum of the components $H^0(C, \mathcal{O}(2p_j + D_j)) \simeq (T_{p_j}C)^{\otimes 2}$ for $j \neq i$. Furthermore, the restriction of (3.3.4) to the subspace $H^0(C, \mathcal{O}(2p_j + D_j))$ is exactly α_{ji} . This immediately implies the assertion. \square

Similarly to Corollary 3.3.2 this leads to the following result.

Corollary 3.3.4. *Let $\text{char}(\mathbb{K}) = 0$ and $g \geq 6$. Then a generic $(C, p_1, \dots, p_g) \in \mathcal{M}_{g,g}$ is uniquely determined by the configuration of g points p_1, \dots, p_g and g tangent lines L_i to C at these points in the embedding given by the linear system $|2(p_1 + \dots + p_g)|$, together with identifications $(T_{p_i}L_i)^2 \simeq \mathcal{O}(1)|_{p_i}$ obtained from (3.3.1).*

Remark 3.3.5. One can ask whether in Corollaries 3.3.2 and 3.3.4 it is enough to consider simply the configuration of points (p_i) and lines (L_i) , for sufficiently large g . We do not know the answer. Note that the number of parameters describe such a configuration is $g^2 - 3g$, so one should take $g \geq 7$ in order for this to have a chance to be true.

Recall that for a line bundle L one has the Wahl map (see [30])

$$W_L : \bigwedge^2 H^0(C, L) \rightarrow H^0(C, L^2 \otimes \omega_C)$$

By definition,

$$W_L(s_1 \wedge s_2)(p) = \varphi'_1(p)\varphi_2(p) - \varphi'_2(p)\varphi_1(p),$$

where φ_1, φ_2 are local functions at p corresponding to s_1, s_2 via some local trivialization of L . In invariant terms, the functional

$$W_{L,p} : \bigwedge^2 H^0(C, L) \rightarrow (L^2 \otimes \omega_C)|_p : s_1 \wedge s_2 \rightarrow W_L(s_1 \wedge s_2)(p)$$

is given by restricting to a neighborhood $U \subset C$ of p and applying the composition

$$\bigwedge^2 H^0(U, L) \rightarrow H^0(U, L(-p)) \otimes L|_p \xrightarrow{\text{ev}_p \otimes \text{id}} L(-p)|_p \otimes L|_p \simeq (L^2 \otimes \omega_C)|_p,$$

where the first map is induced by the exact sequence

$$0 \rightarrow H^0(U, L(-p)) \rightarrow H^0(U, L) \rightarrow L|_p \rightarrow 0.$$

In the case when L is base point free we can take $U = C$, and we see that $W_{L,p}$ is essentially given by the Plücker coordinates of the tangent line to $\varphi_L(C) \subset \mathbb{P}(H^0(C, L)^*)$ at p .

Now let (C, p_1, \dots, p_g) be as before. Given the interpretation of (α_{ij}) in terms of tangent lines from Propositions 3.3.1 and 3.3.3, we can relate it to the Wahl maps W_L associated with $L = \omega_C$ and $L = \mathcal{O}_C(2D)$.

In the case $L = \omega_C$ we have a natural decomposition $H^0(C, \omega_C) = \bigoplus_{i=1}^g T_{p_i}^*$, and for $i \neq j$ the restriction

$$T_{p_i}^{-1} \otimes T_{p_j}^{-1} \hookrightarrow \bigwedge^2 H^0(C, \omega_C) \xrightarrow{W_{\omega_C, p_i}} T_{p_i}^{-3}$$

is given by tensoring with $-\alpha_{ij}$. This completely determines W_{ω_C, p_i} since its restrictions to $T_{p_j}^{-1} \otimes T_{p_k}^{-1} \subset \bigwedge^2 H^0(C, \omega_C)$ are zero for $j \neq i, k \neq i$.

In the case $L = \mathcal{O}_C(2D)$ the maps W_{L, p_i} factor through $\bigwedge^2(H^0(C, L)/\langle 1 \rangle)$, since the section $1 \in H^0(C, L)$ has double zero at p_i . We have a decomposition

$$H^0(C, L)/\langle 1 \rangle = \bigoplus_{i=1}^g H^0(C, \mathcal{O}(2p_i + D_i))/\langle 1 \rangle,$$

and an identification of each summand $H^0(C, \mathcal{O}(2p_i + D_i))/\langle 1 \rangle \simeq T_{p_i}^2$. Now the restriction

$$T_{p_j}^2 \otimes T_{p_i}^2 \hookrightarrow \bigwedge^2 (H^0(C, L)/\langle 1 \rangle) \xrightarrow{W_{L, p_i}} (L^2 \otimes \omega_C)|_{p_i} \simeq T_{p_i}^3$$

is given by α_{ji} . Again, this determines W_{L, p_i} , since its restrictions to $T_{p_j}^2 \otimes T_{p_k}^2$ are zero for $j \neq i, k \neq i$.

4. RECONSTRUCTION OF THE CURVE

In this section (C, p_\bullet) corresponds to a generic point of $\mathcal{M}_{g,g}$. In particular, $h^0(D) = 1$, where $D = p_1 + \dots + p_g$.

4.1. Multiplication map. Consider the line bundle

$$L' = \mathcal{O}_C(2D + p_1) = \mathcal{O}_C(3p_1 + 2(p_2 + \dots + p_g))$$

of degree $2g + 1$ on C .

Lemma 4.1.1. *Let $g \geq 4$. For generic (C, p_\bullet) the curve C is cut out by quadrics in the projective embedding given by $|L'|$.*

Proof. By [7, Thm. 2], this is true provided C is not hyperelliptic and $L' \not\cong \omega_C(x + y + z)$ for any $x, y, z \in C$ (i.e., C has no trisecants in the projective embedding given by $|L'|$). Since L' is determined by $g \geq 4$ points on C , this holds generically. \square

Thus, for $g \geq 4$, generically we can recover the image of C in \mathbb{P}^{g+1} from the multiplication map

$$H^0(C, L') \otimes H^0(C, L') \rightarrow H^0(C, (L')^2). \quad (4.1.1)$$

By Riemann-Roch, we have $h^0(L') = g + 2$, $h^0((L')^2) = 3g + 3$.

Lemma 4.1.2. *For $i = 1, \dots, g$, let us pick a nonconstant rational function $f_i \in H^0(C, \mathcal{O}(D + p_i))$ and a rational function $h_i \in H^0(C, \mathcal{O}(D + 2p_1)) \setminus H^0(C, \mathcal{O}(D + p_1))$. Then we have the following bases in $H^0(C, L')$ and $H^0(C, (L')^2)$:*

$$\begin{aligned} H^0(C, L') : & 1, f_1, \dots, f_g, h_1; \\ H^0(C, (L')^2) : & 1, f_1, \dots, f_g, h_1, \dots, h_g, f_1^2, \dots, f_g^2, f_1 h_1, h_1^2. \end{aligned}$$

Proof. The exact sequences

$$0 \rightarrow H^0(C, \mathcal{O}(nD)) \rightarrow H^0(C, \mathcal{O}((n+1)D)) \rightarrow \bigoplus_{i=1}^g H^0(C, \mathcal{O}((n+1)p_i)/\mathcal{O}(np_i)) \rightarrow 0$$

for $n = 1, 2$ and 3 give us the following bases:

$$\begin{aligned} H^0(C, \mathcal{O}(2D)) &: 1, f_1, \dots, f_g; \\ H^0(C, \mathcal{O}(3D)) &: 1, f_1, \dots, f_g, h_1, \dots, h_g; \\ H^0(C, \mathcal{O}(4D)) &: 1, f_1, \dots, f_g, h_1, \dots, h_g, f_1^2, \dots, f_g^2. \end{aligned}$$

Now the result follows from the exact sequences

$$\begin{aligned} 0 \rightarrow H^0(C, \mathcal{O}(2D)) \rightarrow H^0(C, L') \rightarrow H^0(C, \mathcal{O}(3p_1)/\mathcal{O}(2p_1)) \rightarrow 0 \quad \text{and} \\ 0 \rightarrow H^0(C, \mathcal{O}(4D)) \rightarrow H^0(C, (L')^2) \rightarrow H^0(C, \mathcal{O}(6p_1)/\mathcal{O}(4p_1)) \rightarrow 0. \end{aligned}$$

□

We need convenient formal parameters at the marked points.

Lemma 4.1.3. *Let $\text{char}(\mathbb{k}) = 0$ (resp., $\text{char}(\mathbb{k}) > N$ for some N). Let C be a smooth projective curve of genus g . For any point p and any divisor E of degree $g - 1$ such that $p \notin \text{supp}(E)$ and $h^0(p + E) = 1$ there exists a formal parameter $t_{p,E}$ (resp., formal parameter modulo \mathfrak{m}^{N+1}), unique up to rescaling by a constant, such that for every $n \geq 2$ (resp., for $2 \leq n \leq N$), there exists a global section of $\mathcal{O}(np + E)$ with the polar part $t_{p,E}^{-n}$ at p .*

Proof. Pick a non-constant function $f(2) \in H^0(C, \mathcal{O}(2p + E))$. Then for any local parameter t at p we can rescale $f(2)$ so that

$$f(2) = \frac{1}{t^2} + \frac{c}{t} + \dots$$

at p , where c depends only on $t \bmod \mathfrak{m}^3$. Replacing t by $t + at^2 \bmod \mathfrak{m}^3$ leads to the transformation $c \mapsto c - 2a$. This implies the statement for $n = 2$. Then we proceed by induction: suppose we have a local parameter $t \bmod \mathfrak{m}^n$ and functions $f(m) \in H^0(C, \mathcal{O}(mp + E))$ with polar parts t^{-m} for $2 \leq m \leq n - 1$. Let us take $f(n) \in H^0(C, \mathcal{O}(np + E)) \setminus H^0(C, \mathcal{O}((n-1)p + E))$. Rescaling and subtracting an appropriate linear combination of $f(2), \dots, f(n-1)$ we get a unique such $f(n)$ with

$$f(n) = \frac{1}{t^n} + \frac{c}{t} + \dots$$

at p , where we extend $t \bmod \mathfrak{m}^n$ to $t \bmod \mathfrak{m}^{n+1}$ in some way. Changing t by $t + at^n$ leads to the change of c to $c - na$, so we find the unique $t \bmod \mathfrak{m}^{n+1}$ for which $c = 0$. □

Let $D_i = \sum_{j \neq i} p_j$. Applying Lemma 4.1.3 (for $\text{char}(\mathbb{k}) \neq 2, 3$) to the pairs (p_i, D_i) we can choose formal parameters $(t_i = t_{p_i, D_i})$ at p_i , so that there are elements $f_i \in H^0(C, \mathcal{O}(2p_i + D_i))$, $h_i \in H^0(C, \mathcal{O}(3p_i + D_i))$ and $k_i \in H^0(C, \mathcal{O}(4p_i + D_i))$ for $i = 1, \dots, n$, such that

$$f_i \equiv \frac{1}{t_i^2} \bmod \hat{\mathcal{O}}_{C, p_i},$$

$$h_i \equiv \frac{1}{t_i^3} \bmod \hat{\mathcal{O}}_{C,p_i},$$

$$k_i \equiv \frac{1}{t_i^4} \bmod \hat{\mathcal{O}}_{C,p_i},$$

where $\hat{\mathcal{O}}_{C,p_i}$ is the completion of the local ring \mathcal{O}_{C,p_i} . This fixes f_i , h_i and k_i up to adding a constant. Let p_j be another marked point (so $j \neq i$). We have expansions

$$f_i \equiv \frac{\alpha_{ij}}{t_j} + \delta_{ij} + \eta_{ij}t_j \bmod t_j^2 \hat{\mathcal{O}}_{C,p_i},$$

$$h_i \equiv \frac{\beta_{ij}}{t_j} + \varepsilon_{ij} + \vartheta_{ij}t_j \bmod t_j^2 \hat{\mathcal{O}}_{C,p_i},$$

$$k_i \equiv \frac{\gamma_{ij}}{t_j} + \zeta_{ij} \bmod t_j \hat{\mathcal{O}}_{C,p_i},$$

for some constants (α_{ij}) , (β_{ij}) , (γ_{ij}) , $(\delta_{ij})^3$, (ε_{ij}) , (ζ_{ij}) , (η_{ij}) , (ϑ_{ij}) (defined for $i \neq j$). Note that here (α_{ij}) are the functions defined by (3.1.1) (with some choices of trivializations of the tangent spaces T_{p_i}). Adding a constant to each f_i (resp., h_i , k_i) we can assume that

$$\delta_{i,i+1} = 0, \quad \varepsilon_{i,i+1} = 0, \quad \zeta_{i,i+1} = 0 \quad (4.1.2)$$

for $i = 1, \dots, g$ (where we think of indices as elements of $\mathbb{Z}/g\mathbb{Z}$). This fixes the choice of f_i , h_i and k_i , for $i = 1, \dots, g$, uniquely. Let also define for each $i = 1, \dots, g$ a constant δ_{ii} , so that at p_i we have the expansion

$$f_i \equiv \frac{1}{t_i^2} + \delta_{ii} \bmod t_i \hat{\mathcal{O}}_{C,p_i} \quad (4.1.3)$$

for some constants (δ_{ii}) .

To describe the multiplication map (4.1.1) in terms of the bases of Lemma 4.1.2 we need to find the decompositions of the products $f_i f_j$ for $i \neq j$ and $f_i h_1$ for $i \neq 1$.

Lemma 4.1.4. *(i) For $i \neq j$ one has*

$$f_i f_j = \sum_{k \neq i,j} \alpha_{ik} \alpha_{jk} f_k + \alpha_{ji} h_i + \alpha_{ij} h_j + \delta_{ji} f_i + \delta_{ij} f_j + a_{ij}, \quad (4.1.4)$$

for some constants $a_{ij} = a_{ji}$. Furthermore, one has the following relations:

$$\eta_{ij} + \alpha_{ij} \delta_{jj} = \sum_{k \neq i,j} \alpha_{ij} \alpha_{jk} \alpha_{kj} + \alpha_{ji} \beta_{ij} + \delta_{ji} \alpha_{ij}, \quad (4.1.5)$$

$$\alpha_{ik}(\delta_{jk} - \delta_{ji}) + \alpha_{jk}(\delta_{ik} - \delta_{ij}) = \sum_{l \neq i,j,k} \alpha_{il} \alpha_{jl} \alpha_{lk} + \alpha_{ji} \beta_{ik} + \alpha_{ij} \beta_{jk}, \quad (4.1.6)$$

$$\alpha_{ik} \eta_{jk} + \alpha_{jk} \eta_{ik} + \delta_{ik} \delta_{jk} = \sum_{l \neq i,j,k} \alpha_{il} \alpha_{jl} \delta_{lk} + \alpha_{ji} \varepsilon_{ik} + \alpha_{ij} \varepsilon_{jk} + \delta_{ji} \delta_{ik} + \delta_{ij} \delta_{jk} + a_{ij}, \quad (4.1.7)$$

where i, j, k are distinct.

³We do not use the Kronecker delta in this paper

(ii) For $i \neq j$ one has

$$f_i h_j = \sum_{k \neq i, j} \alpha_{ik} \beta_{jk} f_k + \alpha_{ij} k_j + \beta_{ji} h_i + \delta_{ij} h_j + \varepsilon_{ji} f_i + \eta_{ij} f_j + b_{ij} \quad (4.1.8)$$

for some constants (b_{ij}) . Furthermore, one has the following relations

$$\vartheta_{ji} + \beta_{ji} \delta_{ii} = \sum_{k \neq i, j} \alpha_{ik} \beta_{jk} \alpha_{ki} + \alpha_{ij} \gamma_{ji} + \delta_{ij} \beta_{ji} + \eta_{ij} \alpha_{ji}, \quad (4.1.9)$$

$$\alpha_{ik} \varepsilon_{jk} + \delta_{ik} \beta_{jk} = \sum_{l \neq i, j, k} \alpha_{il} \beta_{jl} \alpha_{lk} + \alpha_{ij} \gamma_{jk} + \beta_{ji} \beta_{ik} + \delta_{ij} \beta_{jk} + \varepsilon_{ji} \alpha_{ik} + \eta_{ij} \alpha_{jk}, \quad (4.1.10)$$

$$\alpha_{ik} \vartheta_{jk} + \delta_{ik} \varepsilon_{jk} + \eta_{ik} \beta_{jk} = \sum_{l \neq i, j} \alpha_{il} \beta_{jl} \delta_{lk} + \alpha_{ij} \zeta_{jk} + \beta_{ji} \varepsilon_{ik} + \delta_{ij} \varepsilon_{jk} + \varepsilon_{ji} \delta_{ik} + \eta_{ij} \delta_{jk} + b_{ij}, \quad (4.1.11)$$

where i, j, k are distinct (note that in right-hand side of the last equation we allow $l = k$ in the sum).

Proof. (i) We have $f_i f_j \in H^0(C, \mathcal{O}(3p_i + 3p_j + 2D_{ij}))$. Expanding in the formal parameter at p_i we obtain

$$f_i f_j = \left(\frac{1}{t_i^2} + \delta_{ii} + \dots \right) \left(\frac{\alpha_{ji}}{t_i} + \varepsilon_{ji} + \eta_{ji} t + \dots \right) = \frac{\alpha_{ji}}{t_i^3} + \frac{\delta_{ji}}{t_i^2} + \frac{\eta_{ji} + \alpha_{ji} \delta_{ii}}{t_i} + \dots \quad (4.1.12)$$

Hence, the difference

$$f_i f_j - \alpha_{ji} h_i - \alpha_{ij} h_j - \delta_{ji} f_i - \delta_{ij} f_j$$

has poles of order at most 1 at p_i and p_j . On the other hand, expanding at p_k , where $k \neq i, j$ we obtain

$$f_i f_j = \frac{\alpha_{ik} \alpha_{jk}}{t_k^2} + \frac{\alpha_{ik} \delta_{jk} + \alpha_{jk} \delta_{ik}}{t_k} + \alpha_{ik} \eta_{jk} + \alpha_{jk} \eta_{ik} + \delta_{ik} \delta_{kj} \bmod t_k \hat{\mathcal{O}}_{C, p_k}. \quad (4.1.13)$$

Hence,

$$f_i f_j - \sum_{k \neq i, j} \alpha_{ik} \alpha_{jk} f_k - \alpha_{ji} h_i - \alpha_{ij} h_j - \delta_{ji} f_i - \delta_{ij} f_j$$

has poles of order at most 1 at all marked points. Since such a function has to be constant, this implies (4.1.4). Now the relation (4.1.5) is obtained by equating polar parts of both sides of (4.1.4) at p_j . Similarly, (4.1.6) and (4.1.7) are obtained by considering expansions of both sides of (4.1.4) at p_k , where $k \neq i, j$.

(ii) The proof of (4.1.8) is similar to that of (4.1.4). Comparing the polar parts of both sides of (4.1.8) at p_i we obtain (4.1.9). The relations (4.1.10) and (4.1.11) are obtained by considering expansions of both sides of (4.1.8) at p_k , where $k \neq i, j$. \square

Proof of Theorem 3.2.1. We would like to prove that the map $\alpha : \mathcal{M}_{g,g}^{(1)} \xrightarrow{(\alpha_{ij})} \mathbb{G}_m^{g^2-g}$ is generically one-to-one on its image for $g \geq 6$. Since the restriction of α to fibers of the projection $\mathcal{M}_{g,g}^{(1)} \rightarrow \mathcal{M}_{g,g}$ is injective, it is enough to show how to recover generic (C, p_\bullet) from the constants (α_{ij}) , defined using some trivializations of the tangent spaces T_{p_i} . By Lemma 4.1.1, for generic (C, p_\bullet) the kernel of the multiplication map (4.1.1) gives quadratic

equations which cut out C in the projective embedding given by $|2D + p_1|$, where $D = p_1 + \dots + p_g$. If in addition we know the line spanned by the section $1 \in H^0(C, \mathcal{O}(2D + p_1))$ then we can recover p_1 as a triple zero of this section and the unordered collection of points p_2, \dots, p_g as double zeros of this section. Furthermore, we can recover each p_i for $i \geq 2$ if we know the line spanned by the section $f_i \in H^0(C, \mathcal{O}(D + p_i)) \subset H^0(C, \mathcal{O}(2D + p_1))$ used in Lemma 4.1.2. Indeed, generically f_i , viewed as a section of L' , is nonzero near p_i and has simple zeros at p_j for $j \neq i$, $j \geq 2$.

By (4.1.4) and (4.1.8), the constants (β_{ij}) , (δ_{ij}) , (ε_{ij}) , (η_{ij}) , (a_{ij}) and (b_{ij}) (where $i \neq j$) determine the multiplication map (4.1.1) with respect to the bases of Lemma 4.1.2. Thus, it is enough to show that for generic $(C, p_\bullet) \in \overline{\mathcal{M}}_{g,g}$ these constants are uniquely determined by (α_{ij}) . We do this by solving the equations obtained in Lemma 4.1.4.

Step 1. We would like to solve the equations (4.1.6) (together with the condition $\delta_{i,i+1} = 0$) for (β_{ij}) , (δ_{ij}) . The fact that for generic (C, p_\bullet) these equations determine (β_{ij}) and (δ_{ij}) follows from Proposition 4.2.2(i) below.

Step 2. Note that we can express η_{ij} in terms of δ_{jj} (and known quantities) using (4.1.5). Now substituting in (4.1.7) we get a linear system for (ε_{ij}) , (a_{ij}) and (δ_{ii}) of the form

$$\alpha_{ji}\varepsilon_{ik} + \alpha_{ij}\varepsilon_{jk} + 2\alpha_{ik}\alpha_{jk}\delta_{kk} + a_{ij} = A_{ijk} \quad (4.1.14)$$

By Proposition 4.2.2(ii) below, for generic (C, p_\bullet) these equations (together with the condition $\varepsilon_{i,i+1} = 0$) determine (ε_{ij}) , (a_{ij}) and (δ_{ii}) uniquely.

Step 3. Since for generic (C, p_\bullet) all α_{ij} are nonzero, the equations (4.1.10) determine (γ_{ij}) uniquely (note that even for $g = 6$ we have more equations than needed).

Step 4. Using (4.1.9) we can express ϑ_{ij} in terms of γ_{ij} . Hence, (4.1.11) can be viewed as the system of equations for (ζ_{ij}) and (b_{ij}) of the form

$$\alpha_{ij}\zeta_{jk} + b_{ij} = B_{ijk}.$$

Generically, $\alpha_{ij} \neq 0$, so we can rewrite this as

$$\zeta_{jk} + \alpha_{ij}^{-1}b_{ij} = B'_{ijk}.$$

These equations together with the condition $\zeta_{i,i+1} = 0$ determine (ζ_{ij}) and (b_{ij}) uniquely. \square

Remark 4.1.5. The above reconstruction procedure gives also a way to produce some polynomial equations for (α_{ij}) for $g \geq 6$. For example, for $g = 6$ the system (4.1.6) of 60 equations has 54 variables (since we set $\delta_{i,i+1} = 0$). Taking any 55 equations we get the vanishing of a 55×55 determinant, with one column of homogeneous cubic polynomials in (α_{ij}) and all other entries linear in (α_{ij}) , which gives a degree-57 equation on (α_{ij}) . One can check that this indeed leads to nonzero equations. Another potential source of equations comes from the interpretation of (α_{ij}) as constants determining m_3 on $E_{g,g}$ (see Proposition 1.3.3 and Corollary 2.4.3). The condition of existence of compatible m_4 should give some equations on (α_{ij}) .

4.2. Degeneration argument. So far, we have reduced our reconstruction problem to proving that certain linear systems have maximal possible rank generically on $\overline{\mathcal{M}}_{g,g}$.

Namely, consider the homogeneous linear system on $(\beta_{ij}, \delta_{ij})$, associated with (4.1.6),

$$\alpha_{ik}(\delta_{jk} - \delta_{ji}) + \alpha_{jk}(\delta_{ik} - \delta_{ij}) = \alpha_{ji}\beta_{ik} + \alpha_{ij}\beta_{jk}, \quad (4.2.1)$$

and the homogeneous linear system on $(\varepsilon_{ij}, a_{ij}, \delta_{ii})$, associated with (4.1.14),

$$\alpha_{ji}\varepsilon_{ik} + \alpha_{ij}\varepsilon_{jk} + 2\alpha_{ik}\alpha_{jk}\delta_{kk} + a_{ij} = 0, \quad (4.2.2)$$

(in both systems i, j, k are distinct). We have to check that generically they have only the obvious solutions

$$\delta_{ij} = \lambda_i, \quad \beta_{ij} = 0, \quad (4.2.3)$$

$$\varepsilon_{ij} = -\mu_i, \quad a_{ij} = \alpha_{ji}\mu_i + \alpha_{ij}\mu_j, \quad \delta_{ii} = 0, \quad (4.2.4)$$

for some (λ_i) and (μ_i) . Our strategy is to reduce this to the case $g = 6$ and to study the above systems for irreducible rational nodal curves, for which α_{ij} can be determined explicitly.

Namely, consider the curve C obtained from \mathbb{P}^1 by gluing g pairs of distinct points $(a_1, b_1), \dots, (a_g, b_g)$, where $a_i, b_i \in \mathbb{A}^1$, together with the marked points $p_1, \dots, p_g \in C$ that are images of the points $c_1, \dots, c_g \in \mathbb{A}^1 \subset \mathbb{P}^1$. Note that the coordinate on \mathbb{A}^1 gives rise to a trivialization of the tangent line to C at each p_i . We look for the rational functions $f_i \in H^0(C, \mathcal{O}(2p_i + D_i))$ in the form

$$f_i(t) = \frac{1}{(t - c_i)^2} + \sum_{j=1}^g \frac{\alpha_{ij}}{t - c_j},$$

where for $i \neq j$ the constants α_{ij} are the functions we are interested in (while α_{ii} do not have an invariant meaning). Note that to compute α_{ij} we do need the special choice of parameters at p_i that we used for Lemma 4.1.4. The conditions $f_i(a_k) = f_i(b_k)$ give the following system of linear equations on (α_{ij}) :

$$\sum_{j=1}^g \left(\frac{1}{b_k - c_j} - \frac{1}{a_k - c_j} \right) \alpha_{ij} = \frac{1}{(a_k - c_i)^2} - \frac{1}{(b_k - c_i)^2}, \quad 1 \leq i, k \leq g.$$

Dividing by $a_k - b_k$ we can rewrite this as

$$\sum_{j=1}^g \frac{1}{(b_k - c_j)(a_k - c_j)} \alpha_{ij} = \frac{2c_i - a_k - b_k}{(a_k - c_i)^2(b_k - c_i)^2}, \quad 1 \leq i, k \leq g. \quad (4.2.5)$$

Let us consider the $g \times g$ -matrices $A = (\alpha_{ij})$, $M = (m_{ij})$ and $N = (n_{ij})$, where

$$m_{ij} = \frac{1}{(b_j - c_i)(a_j - c_i)}, \quad n_{ij} = \frac{2c_i - a_j - b_j}{(a_j - c_i)^2(b_j - c_i)^2}. \quad (4.2.6)$$

Then (4.2.5) is simply the matrix relation

$$AM = N.$$

Note that the matrices M and N are also defined when $a_i = b_i$.

Lemma 4.2.1. *Let $\text{char}(\mathbb{k}) = 0$. For $g = 6$ and*

$$a_i = b_i = -c_i = i \text{ for } i = 1, \dots, 6,$$

the matrix M is invertible. Furthermore, for the corresponding entries α_{ij} of the matrix $A = NM^{-1}$ each of the systems (4.2.1) and (4.2.2) has 6 free variables. Hence, the same assertion is true for generic a_i, b_i, c_i .

Proof. We checked this with the help of the computer (see Appendix). \square

Proposition 4.2.2. *Let $\text{char}(\mathbb{k}) = 0$ and $g \geq 6$.*

(i) At generic point of $\overline{\mathcal{M}}_{g,g}^{(1)}$ the system (4.2.1) has only trivial solutions (4.2.3).

(ii) At generic point of $\overline{\mathcal{M}}_{g,g}^{(1)}$ the system (4.2.2) has only trivial solutions (4.2.4).

Proof. (i) Lemma 4.2.1 implies that the assertion is true for generic $(C, p_1, \dots, p_6) \in \mathcal{M}_{6,6}^{(1)}$.

For $g > 6$ let us fix a subset $I \subset \{1, \dots, g\}$ consisting of 6 elements. We claim that generically the only solution of the equations (4.2.1) with $i, j, k \in I$ for variables $(\beta_{ij}, \delta_{ij} \mid i, j \in I)$ is

$$\beta_{ij} = 0, \quad \delta_{ij} = \lambda_{i,I}, \quad (4.2.7)$$

for some constants $\lambda_{i,I}$. Indeed, without loss of generality we can assume that $I = \{1, \dots, 6\}$. Let us take generic curves $(C_1, p_1, \dots, p_6) \in \mathcal{M}_{6,6}$ and $(C_2, p_7, \dots, p_g) \in \mathcal{M}_{g-6, g-6}$ and consider the nodal curve (C, p_1, \dots, p_g) obtained from $C_1 \sqcup C_2$ by identifying points $p \in C_1$ and $q \in C_2$ (where p and q are different from all the markings). We also assume that nonzero tangent vector fields are chosen at all points, so α_{ij} are defined for $i \neq j$. Now for $i \in \{1, \dots, 6\}$ a nonconstant section $f_i \in H^0(C, \mathcal{O}(2p_i + D_i))$ will restrict to a similar section on C_1 (and will have a constant restriction to C_2). Hence for $i, j \in \{1, \dots, 6\}$ the constants α_{ij} calculated for (C, p_\bullet) are equal to those for (C_1, p_1, \dots, p_6) . This implies our claim.

Thus, for generic (C, p_\bullet) a solution of (4.1.6) satisfies (4.2.7) for each I as above. Note that for I and I' such that $i, j \in I \cap I'$ we have $\lambda_{i,I} = \lambda_{i,I'} = \delta_{ij}$. Since any two subsets I containing i can be connected by a chain of subsets containing i , in which every two consecutive terms have at least two elements in common, this implies that $\lambda_{i,I}$ depends only on i .

(ii) For $g = 6$ this follows from Lemma 4.2.1. Then we proceed as in part (i). \square

5. THE TANGENT MAP

5.1. General formula. It is well known that the tangent space to the moduli space $\mathcal{M}_{g,g}^{(1)}$ at a stable curve $(C, p_1, \dots, p_g, v_1, \dots, v_g)$ (where $v_i \in T_{p_i} \setminus \{0\}$) is canonically identified with $\text{Ext}^1(\Omega_C, \mathcal{O}(-2D))$, where Ω_C denotes the sheaf of Kähler differentials and $D = \sum_{i=1}^g p_i$. On the other hand, if $h^0(\mathcal{O}(D)) = 1$ then we can use the boundary homomorphism

$$\bigoplus_{j \neq i} T_{p_j} \simeq H^0(C, \mathcal{O}(D)/\mathcal{O}(p_i)) \xrightarrow{\sim} H^1(C, \mathcal{O}(p_i))$$

to get natural bases in each space $H^1(C, \mathcal{O}(p_i))$ numbered by e_{ij} , $i \neq j$.

Let us consider the regular map

$$\alpha^{reg} : \mathcal{U}^{(1)} \xrightarrow{(\alpha_{ij})} \mathbb{A}^{g^2-g},$$

where $\mathcal{U}^{(1)} \subset \overline{\mathcal{M}}_{g,g}^{(1)}$ is the preimage of the open substack $\mathcal{U} \subset \overline{\mathcal{M}}_{g,g}$ defined by $h^0(D) = 1$.

Proposition 5.1.1. *Under the above identifications the tangent map to α^{reg} is the map*

$$\mathrm{Ext}^1(\Omega_C, \mathcal{O}(-2D)) \xrightarrow{(df_i)} \bigoplus_{i=1}^g \mathrm{Ext}^1(\mathcal{O}(-2D - p_i), \mathcal{O}(-2D)) \simeq \bigoplus_{i=1}^g H^1(C, \mathcal{O}(p_i)), \quad (5.1.1)$$

where for each i , $df_i \in \Omega_C(2D + p_i)$ is the differential of the rational function $f_i \in H^0(\mathcal{O}(D + p_i))$, such that $f_i \equiv v_i^2 \bmod \mathcal{O}(D)$.

Proof. By irreducibility of the moduli space it is enough to consider the case when C is smooth. Then (5.1.1) is the map induced on H^1 by the morphism of coherent sheaves

$$\mathcal{T}(-2D) \xrightarrow{(df_i)} \bigoplus_i \mathcal{O}(p_i).$$

Note also that for each i the natural map

$$\mathcal{O}(p_i) \rightarrow \bigoplus_{j \neq i} \mathcal{O}(D_j)$$

induces an isomorphism on H^1 (recall that $D_j = D - p_j$). Hence, our assertion reduces to checking that for each $i \neq j$, the differential $d\alpha_{ij}$ of the function α_{ij} at $(C, p_1, \dots, p_g, v_1, \dots, v_g)$ is equal to the map induced on H^1 by the morphism

$$\mathcal{T}(-2D) \xrightarrow{df_i} \mathcal{O}(p_i) \rightarrow \mathcal{O}(D_j). \quad (5.1.2)$$

To this end let us fix an affine covering (U_a) of C and a Čech 1-cocycle v_{ab} with values in $\mathcal{T}(-2D)$ (we assume that each marked point is contained in only one U_a). This gives a first-order deformation of (C, p_1, \dots, p_g) over $\mathbb{k}[\epsilon]/(\epsilon^2)$, glued from the trivial deformations $U_a[\epsilon] := U_a \times \mathrm{Spec}(\mathbb{k}[\epsilon]/(\epsilon^2))$ of U_a with the transitions on $U_{ab}[\epsilon]$ given by the automorphisms $\mathrm{id} + \epsilon v_{ab}$. Let $f_i \in H^0(C, \mathcal{O}(D + p_i))$ be such that $f_i \equiv v_i^2 \bmod \mathcal{O}(D)$, so that by definition

$$f_i \equiv \alpha_{ij} \cdot v_j \bmod \mathcal{O}(D_j + p_i).$$

We want to deform this function over $\mathbb{k}[\epsilon]/(\epsilon^2)$ preserving the condition $f_i \equiv v_i^2 \bmod \mathcal{O}(D)$. Thus, the deformed function should have form $f_i + \epsilon g_a$ on each U_a , where $g_a \in H^0(U_a, \mathcal{O}(D))$. The gluing condition gives

$$f_i + \epsilon g_b = (\mathrm{id} + \epsilon v_{ab})(f_i + \epsilon g_a)$$

on $U_{ab}[\epsilon]$, which is equivalent to

$$g_b = g_a + v_{ab}(f_i). \quad (5.1.3)$$

Then $d\alpha_{ij}$ is the image of $g_{a(j)}$ in $\mathcal{O}(p_j)/\mathcal{O}$, where $p_j \in U_{a(j)}$. From the exact sequence

$$0 \rightarrow \mathcal{O}(D_j) \rightarrow \mathcal{O}(D) \rightarrow \mathcal{O}(p_j)/\mathcal{O} \rightarrow 0$$

we get the following exact sequence of Čech complexes:

$$\begin{array}{ccccccc}
0 \rightarrow & C^0(\mathcal{O}(D_j)) & \rightarrow & C^0(\mathcal{O}(D)) & \xrightarrow{r_j} & C^0(\mathcal{O}(p_j)/\mathcal{O}) & \rightarrow 0 \\
& \downarrow & & \downarrow \delta & & \downarrow & \\
0 \rightarrow & C^1(\mathcal{O}(D_j)) & \xrightarrow{\iota} & C^1(\mathcal{O}(D)) & \longrightarrow & 0 & \longrightarrow 0
\end{array}$$

Note that

$$v_{ab}(f_i) = \langle v_{ab}, df_i \rangle \in \mathcal{O}(p_i) \subset \mathcal{O}(D_j),$$

so we can view $(v_{ab}(f_i))$ as a 1-cocycle in $C^1(\mathcal{O}(D_j))$. By (5.1.3), the cochain $(g_a) \in C^0(\mathcal{O}(D))$ satisfies

$$\delta((g_a)) = \iota(v_{ab}(f_i)).$$

Since $r_j((g_a))$ is exactly $d\alpha_{ij}$, we obtain that the class $[v_{ab}(f_i)] \in H^1(C, \mathcal{O}(D_j))$ is the image of $d\alpha_{ij}$ under the connecting homomorphism $H^0(\mathcal{O}(p_j)/\mathcal{O}) \rightarrow H^1(\mathcal{O}(D_j))$. Since the map (5.1.2) is given by $v \mapsto v(f_i)$, this implies our claim. \square

5.2. Tangent map at a rational irreducible nodal curve. Let (C, p_1, \dots, p_g) be a stable curve. Using Serre duality we can identify the dual to the tangent map (5.1.1) with the map

$$\bigoplus_{i=1}^g H^0(C, \omega_C(-p_i)) \xrightarrow{(df_i)} H^0(C, \Omega_C \otimes \omega_C(2D)), \quad (5.2.1)$$

where ω_C is the dualizing sheaf on C .

We are going to describe the map (5.2.1) explicitly in the case of the curve C obtained from \mathbb{P}^1 by gluing g pairs of distinct points $(a_1, b_1), \dots, (a_g, b_g)$, with the marked points c_1, \dots, c_g . Let us denote by $q_i \in C$ the node corresponding to a pair (a_i, b_i) . Recall that in this case the functions $f_i \in \mathcal{O}(2p_i + D_i)$ correspond to the functions on \mathbb{P}^1

$$f_i(t) = \frac{1}{(t - c_i)^2} + \sum_{j=1}^g \frac{\alpha_{ij}}{t - c_j},$$

where the matrix (α_{ij}) is determined by the conditions $f_i(a_k) = f_i(b_k)$ (see Section 4.2).

The main problem is to understand the space $H^0(C, \Omega_C \otimes \omega_C(2D))$. Recall (see [10, Exer. 5.9]) that Ω_C fits into the exact sequence

$$0 \rightarrow \mathcal{O}_Z \rightarrow \Omega_C \xrightarrow{\nu} \omega_C \rightarrow \mathcal{O}_Z \rightarrow 0,$$

where Z is the union of all nodes. Formally at a node $q \in C$ the curve looks as $\text{Spec}(\mathbb{k}[[x, y]]/(xy))$ and the completion of Ω_C is generated by dx and dy with the relation $x dy = -y dx$, so that the embedding $\mathcal{O}_Z \rightarrow \Omega_C$ is given by $1 \mapsto y dx$. The dualizing sheaf ω_C is locally free and is generated near the node by $dx/x = -dy/y$. Thus, the

space $(\Omega_C \otimes \omega_C) \otimes \mathcal{O}_{C,q}/\mathfrak{m}_q^2$ has the basis $dx \otimes (dx/x), dy \otimes (dx/x), ydx \otimes (dx/x), xdx \otimes (dx/x), ydy \otimes (dx/x)$. Let

$$\tau_q : (\Omega_C \otimes \omega_C) \otimes \mathcal{O}_{C,q}/\mathfrak{m}_q^2 \rightarrow \mathbb{K} \cdot (xdy \otimes \frac{dx}{x})$$

denote the projection to $(xdy \otimes (dx/x))$ with respect to this basis. Then we have an embedding

$$H^0(C, \Omega_C \otimes \omega_C(2D)) \xrightarrow{\nu, (\tau_q)} H^0(C, \omega_C^{\otimes 2}(2D)) \oplus \mathbb{K}^Z. \quad (5.2.2)$$

Thus, to describe (5.2.1) it is enough to consider its compositions with ν and with τ_q for all $q \in Z$.

Let $\pi : \mathbb{P}^1 \rightarrow C$ be the normalization map. Then we have an isomorphism

$$\pi^* \omega_C \simeq \omega_{\mathbb{P}^1} \left(\sum_i (a_i + b_i) \right),$$

so that near the i th node q_i the sections of ω_C are distinguished by the condition $\text{Res}_{a_i} + \text{Res}_{b_i} = 0$. In particular, we have an embedding

$$H^0(C, \omega_C^{\otimes 2}(2D)) \hookrightarrow H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2}(2D + 2 \sum_i (a_i + b_i))).$$

Since the pull-backs $\pi^* df_i \in \omega_{\mathbb{P}^1}(p_i + 2D)$ are regular at all a_i 's and b_i 's, we have a commutative diagram

$$\begin{array}{ccc} \bigoplus_{i=1}^g H^0(C, \omega_C(-p_i)) & \xrightarrow{\nu \circ (df_i)} & H^0(C, \omega_C^{\otimes 2}(2D)) \\ \varphi \downarrow & & \downarrow \\ H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2}(2D + \sum_j (a_j + b_j))) & \longrightarrow & H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2}(2D + 2 \sum_j (a_j + b_j))) \end{array}$$

where φ is induced by the embeddings

$$H^0(C, \omega_C(-p_i)) \hookrightarrow H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^1(-c_i + \sum_j (a_j + b_j)))$$

followed by the product with $\pi^* df_i$. Finally, since $\omega_{\mathbb{P}^1}^{\otimes 2}$ has no global sections, we have an embedding

$$\iota : H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2}(2D + \sum_j (a_j + b_j))) \rightarrow \bigoplus_{i=1}^g \omega_{\mathbb{P}^1}^{\otimes 2}(2c_i)/\omega_{\mathbb{P}^1}^{\otimes 2} \oplus \bigoplus_{j=1}^g \omega_{\mathbb{P}^1}^{\otimes 2}(a_j)/\omega_{\mathbb{P}^1}^{\otimes 2} \oplus \omega_{\mathbb{P}^1}^{\otimes 2}(b_j)/\omega_{\mathbb{P}^1},$$

given by the polar parts at all points a_i, b_i and c_i . Thus, the map (5.2.1) is essentially determined by the map

$$\bigoplus_{i=1}^g H^0(C, \omega_C(-p_i)) \xrightarrow{\iota \circ \varphi, (\tau_q)} \bigoplus_{i=1}^g \omega_{\mathbb{P}^1}^{\otimes 2}(2c_i)/\omega_{\mathbb{P}^1}^{\otimes 2} \oplus \bigoplus_{j=1}^g \omega_{\mathbb{P}^1}^{\otimes 2}(a_j)/\omega_{\mathbb{P}^1}^{\otimes 2} \oplus \omega_{\mathbb{P}^1}^{\otimes 2}(b_j)/\omega_{\mathbb{P}^1} \oplus \mathbb{K}^Z. \quad (5.2.3)$$

In particular, (5.2.3) has the same rank as (5.2.1), which is the same as the rank of the tangent map (5.1.1).

Now we proceed to calculating the components of (5.2.3) explicitly. First, let us describe a basis in the source space. We realize global sections of ω_C as rational 1-forms on \mathbb{P}^1 having 1st order poles at a_i and b_i with opposite residues at a_i and b_i for each i (and regular elsewhere). All such 1-forms on \mathbb{P}^1 can be written as

$$\sum_{k=1}^g \frac{x_k}{(t-a_k)(t-b_k)} dt,$$

for some constants x_1, \dots, x_g . Thus if $(x_{ij}) = M^{-1}$, where $M = (m_{ij})$ (see (4.2.6)), then for each j the form

$$\eta_j = e_j(t)dt, \text{ where } e_j(t) = \sum_{k=1}^g \frac{x_{kj}}{(t-a_k)(t-b_k)}, \quad (5.2.4)$$

is a generator of the 1-dimensional subspace $H^0(C, \omega_C(-D_j)) \subset H^0(C, \omega_C)$. Thus, we can take $(\eta_j)_{j \neq i}$ as a basis of $H^0(C, \omega_C(-p_i))$. It remains to calculate the polar parts of $df_i \otimes \eta_j$, where $i \neq j$, at all the points a_k, b_k and c_k , as well as the constants $\tau_{qk}(df_i \otimes \eta_j)$. The latter constants can be computed as follows.

Lemma 5.2.1. *Let U be a neighborhood of a node $q \in C$, and let x and y be formal parameters at the two points a and b over the node on the normalization. For $\eta \in H^0(U, \omega_U)$ consider the expansions near a and b of its pull-back to the normalization,*

$$\tilde{\eta} = (d_{-1} + d_0x + \dots) \frac{dx}{x}, \quad \tilde{\eta} = (e_{-1} + e_0y + \dots) \frac{dy}{y},$$

where $d_{-1} + e_{-1} = 0$. Then for $f \in \mathcal{O}(U)$ we have

$$\tau_q(df \otimes \eta) = \left(e_0 \frac{d\tilde{f}}{dx}(a) + d_0 \frac{d\tilde{f}}{dy}(b) \right) \cdot xdy \otimes \frac{dx}{x},$$

where \tilde{f} is the pull-back of f to the normalization.

Proof. Let $\tilde{f} = P(x)$ at a and $\tilde{f} = Q(y)$ at b , where $P \in \mathbb{k}[[x]]$, $Q \in \mathbb{k}[[y]]$, $P(0) = Q(0)$. Then

$$df = P'(x)dx + Q'(y)dy \in \Omega_{C,q} \otimes \hat{\mathcal{O}}_{C,q}.$$

Under the trivialization of ω_C in the formal neighborhood q given by dx/x , η corresponds to

$$(d_{-1} + d_0x + d_1x + \dots - e_0y - e_1y - \dots) \frac{dx}{x}.$$

Hence

$$df \otimes \eta = (d_{-1} + d_0x - e_0y + \dots)(P'(x)dx + Q'(y)dy) \otimes \frac{dx}{x}.$$

The terms contributing to τ_q are

$$(d_0Q'(0)xdy - e_0P'(0)ydx) \otimes \frac{dx}{x} = (d_0Q'(0) + e_0P'(0))ydx \otimes \frac{dx}{x}$$

which gives the result. □

To apply this lemma in our case we use expansions of η_j near a_k and b_k :

$$\eta_j = \left(\frac{x_{kj}}{(a_k - b_k)(t - a_k)} + \left[\sum_{l \neq k} \frac{x_{lj}}{(a_k - a_l)(a_k - b_l)} - \frac{x_{kj}}{(a_k - b_k)^2} \right] + \dots \right) dt,$$

$$\eta_j = \left(\frac{x_{kj}}{(b_k - a_k)(t - b_k)} + \left[\sum_{l \neq k} \frac{x_{lj}}{(b_k - a_l)(b_k - b_l)} - \frac{x_{kj}}{(a_k - b_k)^2} \right] + \dots \right) dt.$$

Hence,

$$\tau_{q_k}(df_i \otimes \eta_j) = \tilde{e}_{jk}(a_k) \cdot f'_i(b_k) + \tilde{e}_{jk}(b_k) \cdot f'_i(a_k),$$

where

$$\tilde{e}_{jk}(t) = \sum_{l \neq k} \frac{x_{lj}}{(t - a_l)(t - b_l)} - \frac{x_{kj}}{(a_k - b_k)^2}.$$

Calculation of the polar parts is straightforward. The polar part of $df_i \otimes \eta_j$ at a_k (resp., b_k) is

$$\frac{x_{kj}f'_i(a_k)}{a_k - b_k} \cdot \frac{dt^{\otimes 2}}{t - a_k} \quad \left(\text{resp., } \frac{x_{kj}f'_i(a_k)}{a_k - b_k} \cdot \frac{dt^{\otimes 2}}{t - a_k} \right).$$

To calculate polar parts at c_k we need expansions of f_i and η_j in $t - c_k$, so these will be expressed in terms of α_{ik} and of first two derivatives of $e_j(t)$ at c_k (see (5.2.4)). Namely for $k \neq i, j$ the polar part of $df_i \otimes \eta_j$ at c_k is

$$\frac{e'_j(c_k)\alpha_{ik}}{t - c_k}(dt)^{\otimes 2}$$

The polar part of $df_i \otimes \eta_j$ at c_i is

$$\left(\frac{2e'_j(c_i)}{(t - c_i)^2} + \frac{e''_j(c_i) + e'_j(c_i)\alpha_{ii}}{t - c_i} \right) dt^{\otimes 2}.$$

Finally, the polar part of $df_i \otimes \eta_j$ at c_j is given by

$$\left(\frac{\alpha_{ij}}{(t - c_j)^2} + \frac{e'_j(c_j)\alpha_{ij}}{t - c_j} \right) dt^{\otimes 2}.$$

Theorem 5.2.2. *Assume $\text{char}(\mathbb{k}) = 0$. For $g \leq 5$ the rational map $\bar{\alpha} : \mathcal{M}_{g,g} \rightarrow \mathbb{G}_m^{g^2-2g}$ is dominant.*

Proof. For $g = 3$ this follows from Proposition 3.2.2, which in this case states that the restriction of $\bar{\alpha}$ to the generic fiber of the projection $\mathcal{M}_{g,g} \rightarrow \mathcal{M}_g$ is generically étale. For $g = 4$ and $g = 5$ we use the above calculation to construct a rational irreducible nodal curve with g points for which the tangent map to α (and hence to $\bar{\alpha}$) is surjective. Namely, we check using the computer that for $a_i = -c_i = i$, $b_i = g + i$, where $g = 4$ or 5 , the rank of the map (5.2.3) is $g^2 - g$ (see the GAP codes in the Appendix), hence the tangent map (5.1.1) has the same rank. \square

Remark 5.2.3. By Proposition 3.2.2, in the case $g = 3$ the map $\bar{\alpha}$ is still dominant and generically smooth for $\text{char}(\mathbb{k}) > 0$. In the cases $g = 4$ and $g = 5$ the same is true if $\text{char}(\mathbb{k})$ is sufficiently large.

APPENDIX. GAP codes

1. GAP codes for Lemma 4.2.1.

Setting up vectors $a = (a_i)$, $b = (b_i)$, $c = (c_i)$ and calculating the matrix $A = (\alpha_{ij})$:

```

g := 6; a := [1..g]; b := a; c := -a;
M := NullMat(g, g);
for i in [1..g] do
  for j in [1..g] do
    M[i][j] := 1/((a[j] - c[i]) * (b[j] - c[i])); od; od;
N := NullMat(g, g);
for i in [1..g] do
  for j in [1..g] do
    N[i][j] := (2 * c[i] - a[j] - b[j])/((a[j] - c[i])^2 * (b[j] - c[i])^2); od; od;
A := N/M;

```

Calculating the number of free variables in the system (4.2.1), where we write the coefficients of each equation in a $2g \times g$ -matrix, with one block corresponding to the variables (δ_{ij}) and the other to the variables (β_{ij}) :

```

T := Tuples([1..g], 3);
for S in Tuples([1..g], 3) do
  if S[1] >= S[2] or S[1] = S[3] or S[2] = S[3] then
    RemoveSet(T, S); fi; od;
equations := [];
for S in T do
  m := NullMat(2 * g, g);
  m[S[1] + g][S[3]] := A[S[2]][S[1]]; m[S[2] + g][S[3]] := A[S[1]][S[2]];
  m[S[2]][S[3]] := -A[S[1]][S[3]]; m[S[2]][S[1]] := A[S[1]][S[3]];
  m[S[1]][S[3]] := -A[S[2]][S[3]]; m[S[1]][S[2]] := A[S[2]][S[3]];
  Add(equations, m); od;
V := FreeLeftModule(Rationals, equations);
2 * g * (g - 1) - Dimension(V);

```

Calculating the number of free variables in the system (4.2.2), where we write the coefficients of each equation in a $3g \times g$ -matrix, with blocks corresponding to the variables (δ_{ii}) , (a_{ij}) and (ε_{ij}) , respectively:

```

equations2 := [];
for S in T do
  m := NullMat(3 * g, g);
  m[S[1] + 2 * g][S[3]] := A[S[2]][S[1]]; m[S[2] + 2 * g][S[3]] := A[S[1]][S[2]];
  m[S[1] + g][S[2]] := 1; m[S[3]][S[3]] := 2 * A[S[1]][S[3]] * A[S[2]][S[3]];
  Add(equations2, m); od;
V := FreeLeftModule(Rationals, equations2);
3 * g * (g - 1)/2 + g - Dimension(V);

```

2. GAP codes for Theorem 5.2.2.

Setting up (say, for genus 5) and calculating matrices M , N , A as before, as well as some auxiliary quantities, namely, the matrices $ecp = (e'_j(c_i))$, $ecpp = (e''_j(c_i))$, $fpa = (f'_i(a_j))$, $fpb = (f'_i(b_j))$, $eta = (\tilde{e}_{ji}(a_i))$ and $etb = (\tilde{e}_{ji}(b_i))$:

```

g := 5; a := [1..g]; b := [(g + 1)..(2 * g)]; c := -a;
M := NullMat(g, g); N := NullMat(g, g); Np := NullMat(g, g);
ac2 := NullMat(g, g); ac3 := NullMat(g, g); bc2 := NullMat(g, g); bc3 := NullMat(g, g);
for i in [1..g] do
  for j in [1..g] do
    M[i][j] := (a[j] - c[i])(-1) * (b[j] - c[i])(-1);
    N[i][j] := (2 * c[i] - a[j] - b[j]) * (a[j] - c[i])(-2) * (b[j] - c[i])(-2);
    Np[i][j] := 2 * ((c[i] - a[j])(-3) - (c[i] - b[j])(-3)) / (a[j] - b[j]);
    ac2[i][j] := (a[j] - c[i])(-2); ac3[i][j] := (a[j] - c[i])(-3);
    bc2[i][j] := (b[j] - c[i])(-2); bc3[i][j] := (b[j] - c[i])(-3); od; od;
  A := N/M;
  x := Inverse(M); ecp := -N * x; ecpp := Np * x;
  fpa := 2 * ac3 + A * ac2; fpb := 2 * bc3 + A * bc2;
  eb := NullMat(g, g); ea := NullMat(g, g);
  for i in [1..g] do
    for j in [1..g] do
      if i = j then eb[i][i] := -(b[i] - a[i])(-2); ea[i][i] := eb[i][i];
      else eb[i][j] := (b[i] - a[j])(-1) * (b[i] - b[j])(-1);
      ea[i][j] := (a[i] - a[j])(-1) * (a[i] - b[j])(-1); fi; od; od;
    etb := eb * x; eta := ea * x;

```

In the main cycle we create the set of vectors of length $5g$ numbered by pairs (i, j) , $i \neq j$, representing images of $\eta_j \in H^0(C, \omega_C(-p_i))$ under (5.2.3). The coordinates of these vectors are partitioned into 5 segments of length g (named tau , pa , pb , $pc1$ and $pc2$), corresponding respectively to $\tau_{q_k}(df_i \otimes \eta_j)$, and the polar parts of the Laurent expansions of $df_i \otimes \eta_j$ at a_k , b_k and c_k (the latter are recorded in two segments: coefficients of $(dt)^{\otimes 2}t - c_k$ in positions $[3g + 1, \dots, 4g]$ and coefficients of $(dt)^{\otimes 2}(t - c_k)^2$). The output is the rank of the map (5.2.3).


```

functionals := []; for i in [1..g] do
  for j in [1..g] do
    if i <> j then
      tau := 0 * [1..g]; pa := 0 * [1..g]; pb := 0 * [1..g]; pc1 := 0 * [1..g] pc2 := 0 * [1..g];
      xi := 0 * [1..(5 * g)];
      for k in [1..g] do
        tau[k] := etb[k][j] * fpa[i][k] + eta[k][j] * fpb[i][k];
        pa[k] := x[k][j] * fpa[i][k]; pb[k] := x[k][j] * fpb[i][k];
        if k = i then pc1[i] := ecpp[i][j] + ecp[i][j] * A[i][i]; pc2[i] := 2 * ecp[i][j];
        elif k = j then pc1[j] := ecp[j][j] * A[i][j]; pc2[j] := A[i][j];
        else pc1[k] := ecp[k][j] * A[i][k]; fi;
        xi[k] := tau[k]; xi[k + g] := pa[k]; xi[(k + 2 * g)] := pb[k];
        xi[(k + 3 * g)] := pc1[k]; xi[(k + 4 * g)] := pc2[k]; od ;
      Add(functionals, xi);
    fi; od; od;
  V := FreeLeftModule(Rationals, functionals);
  Dimension(V);

```

For $g = 4$ and $g = 5$ we get the rank equal to 12 and 20, respectively. As a sanity check, for $g = 6$ we get the rank equal to $27 = 5g - 3$ which is the dimension of the moduli space $\mathcal{M}_{6,6}^{(1)}$, which agrees with the fact that for $g \geq 6$ the tangent map is generically injective.

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